RSFT functors for strong cobordisms and finite algebraic torsion

Zhengyi Zhou
Morningside Center of Mathematics
Chinese Academy of Sciences

SFT X, Berlin

Partially joint with Agustin Moreno (Universität Heidelberg)

Computation of RSFT invariants

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Two more functorial properties in this talk:

1. Behavior of APT, P in strong cobordisms.
2. Curves with constraints in exact cobordisms, typically cobordisms from contact $(+1)$ surgeries.

Main theorems–strong cobordisms

Theorem (Moreno-Z. '23)

Let W be a strong cobordism from Y_- to Y_+, if $\mathrm{APT}(Y_+) < \infty$, then $\mathrm{APT}(Y_-) < \infty$.

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Remark

The analogous statement for P is not so neat and will appear later.

Main theorems–surgery cobordisms

Theorem (Z. 23)

Let $Y := \partial(V \times \mathbb{D})$ *and* Λ *a Legendrian sphere, if* $[\Lambda] \neq 0 \in H_*(V \times \mathbb{D}; \mathbb{Q})$*, then* Y_{Λ^+} *obtained from a contact* $(+1)$ *surgery along* Λ *has vanishing contact homology.*

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Corollary

Let $S \subset V$ *be a Lagrangian sphere such that* $[S] \neq 0 \in H_*(V; \mathbb{Q})$, *then* $\mathrm{OB}(V, \phi_{\mathrm{DS}}^{-1})$ *has vanishing contact homology, where* ϕ_{DS} *is a Dehn-Seidel twist.*

Corollary (Bourgeois and van Koert)

Overtwisted contact manifolds have vanishing contact homology.

Main theorems–the combination

Latschev and Wendl found contact 3-folds with algebraic torsion k for any k using planar torsion, and conjectured about the higher dimensional case.

Theorem (Z. in progress)

For any $k \in \mathbb{N}$, there exist spinal open books with a planar vertebra in any dimension $\geqslant 5$, such that the algebraic planar torsion is k. Same for algebraic torsion if $\dim \geqslant 7$ assuming the foundation of full SFT as IBL_∞ algebra is established to the level of current contact homology.

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Remark

Our curve mechanism is different, as our curves are from the vertebra/spine, while curves in planar torsion are from "pages".

Functoriality in strong cobordisms

Strong cobordism

- Y is Weinstein cobordant to an OT 3-fold, then Y is OT (Colin, Wand);
- If Y has planar torsion, then Y is strongly cobordant to an OT 3-fold (Wendl).
  - $\mathrm{APT}, \mathrm{AT}$ are not functorial for strong cobordisms.

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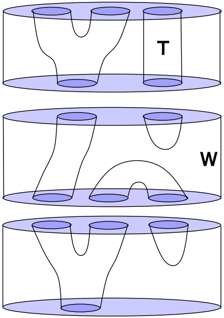
The usual functoriality fails:



Maurer-Cartan elements of BL_∞ algebras

Definition (MC elements)

$\mathfrak{mc} \in \overline{SV}$ of degree 0, s.t. $\widehat{p}(e^{\mathfrak{mc}}) = 0$, where $e^{\mathfrak{mc}} = \sum_{i=1}^{\infty} \frac{\odot^i \mathfrak{mc}}{i!} \in \overline{EV}$.

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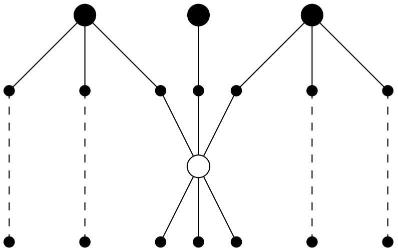

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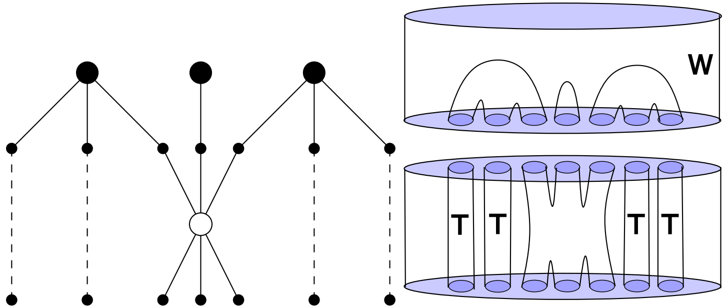
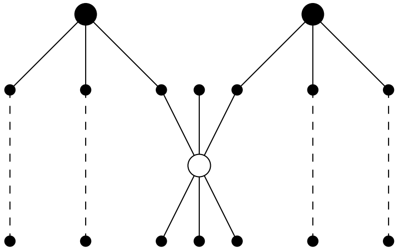


Counting rational holomorphic curves without positive punctures in a strong cobordism $W \Rightarrow$ a MC element $\mathfrak{mc}_W$.

Deformation by MC elements

Given a MC element $\mathfrak{mc}$,

$$p_{\mathfrak{mc}}^{k,l}(v_1 \dots v_k) := \pi_{1,l} \circ \widehat{p}(v_1 \odot \dots \odot v_k \odot e^{\mathfrak{mc}})$$



A component of $p_{\mathfrak{mc}}^{1,7}$

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$$\mathrm{APT}(V_-, p_{-,\mathfrak{mc}_W}) \leqslant \mathrm{APT}(V_+, p_+).$$

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- $\mathrm{id} \odot e^{\mathfrak{mc}_W}$ is a chain map from $(\overline{EV_-}, \widehat{p}_{-,\mathfrak{mc}_W})$ to $(\overline{EV_-}, \widehat{p}_-)$.

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- Assign q_γ with weight $\int \gamma^*\alpha$ and T with weight 1, then $\widehat{p}$ preserves the weight, hence $\widehat{p}_-(v_0) = 1$, where v_0 is the weight 0 part of v.
- $\inf\{\int \gamma^*\alpha\} > 0 \Rightarrow v_0 \in E^k V$, i.e.

$$\mathrm{APT}(V_+, p_+) < +\infty \Rightarrow \mathrm{APT}(V_-, p_-) < +\infty$$

The case for P

Theorem (Moreno-Z. 23)

$\mathrm{P}(Y') < \infty$ *s.t. contributing curves "do not depend on augmentations" (e.g. examples in Agustin's talk). If* $\exists$ *strong cobordism from* Y *to* Y'*,*

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Corollary

For $g \geqslant 1$, $d > 2g - 2$, *the prequantization bundle over* Σ_g *with degree* $-d$ *is not cofillable and has no strong cobordism to* (S^3, ξ_{std}).

Producing algebraic overtwisted contact manifolds

Overtwisted contact sphere

- $\phi_{\mathsf{DS}} \in \pi_0(\mathsf{Symp_c}(D^*S^n))$: the Dehn-Seidel twist;
- $(S^{2n+1}, \xi_{\mathsf{ot}}) = \mathsf{OB}(D^*S^n, \phi_{\mathsf{DS}}^{-1})$: the homotopically standard OT sphere.

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Bourgeois and van Koert showed that, with a suitable contact form and a.c.s., there is a simple Reeb orbit γ winding around the binding once, such that

$$\partial_{\mathrm{CH}}(q_\gamma) = 1,$$

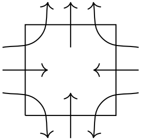
where the holomorphic disk is a “lift" of the natural disk in the binding region.

Via contact connected sum,

$$\mathrm{CH}(S^{2n+1}, \xi_{\mathrm{ot}}) = 0 \Rightarrow \mathrm{CH}(Y_{\mathrm{ot}}) = 0.$$

Contact $(+1)$ surgeries

- $\sum(-x_i\partial_{x_i} + 2y_i\partial_{y_i})$ is a Liouville v.f. on $(D^n_x \times D^n_y, \sum \mathrm{d}x_i \wedge \mathrm{d}y_i)$;



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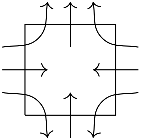

- Both $S^{n-1}_x \times \{0\}$ and $\{0\} \times S^{n-1}_y$ are Legendrians bounding Lagrangian disks $D^n_x \times \{0\}$ and $\{0\} \times D^n_y$;
- Gluing a nbhd of a Legendrian sphere Λ with a nbhd of $S^{n-1}_x \times \{0\}, \{0\} \times S^{n-1}_y$ is called a $-1/+1$ contact surgery, the new contact boundary is $Y_{\Lambda^-}/Y_{\Lambda^+}$.

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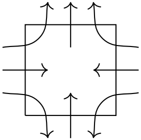

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Fact: $(S^{2n+1}, \xi_{\mathrm{ot}})$ is obtained from applying a $(+1)$ surgery to $\partial(D^*S^n \times \mathbb{D}) = \mathrm{OB}(D^*S^n, \mathrm{id})$ along the Legendrian lift of the zero section $S^n \subset D^*S^n$.
$-1/+1$ surgeries $\Leftrightarrow$ adding positive/negative twists.

No fillings from symplectic cohomology

- Symplectic fillings of $Y := \partial(V \times \mathbb{D})$ have strong unique properties (Eliashberg-Floer-McDuff,Oancea-Viterbo,Barth-Geiges-Zehmisch,Z.);
- For any strong filling W of Y, $\exists x \in SH^*_+(W)$ such that $\delta_\partial(x) = \alpha$, where $\alpha \in H^*(V \times \mathbb{D}) \to H^*(Y)$;

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- $\Rightarrow (S^{2n+1}, \xi_{ot})$ has no strong filling.

From the obstructing curve to the vanishing of CH

The “uniqueness" of filling is from

1. Any closed submanifold $S \subset Y$ such that $\langle \alpha, [S] \rangle \neq 0$, there exist solutions to

$$u : \mathbb{C} \to \mathbb{R} \times Y, \quad \partial_s u + J(\partial_t u - X_H) = 0, \quad \lim_{s \to \infty} u = x, u(0) \in \{0\} \times S.$$

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2. Non-existence of various curves with negative punctures.

- W_Λ: the surgery cobordism from Y_{Λ^+} to Y;
- L: the Lagrangian disk filling of Λ in W_Λ;

By considering holomorphic curves in W_Λ with a point constraint on L and negative punctures, we have

Theorem (Z. 23)

Assume Λ a Legendrian sphere such that $[\Lambda]$ does not vanish in $H_*(V \times \mathbb{D}; \mathbb{Q})$, then $\mathrm{CH}(Y_{\Lambda^+}) = 0$.

Functorial explanation

Motivated by the work of Bourgeois and Oancea,

- try to define SH_+ for a contact manifold, the compactification of

$$\left\{ u : \mathbb{R} \times S^1 \to \widehat{Y}, \quad \partial_s u + J(\partial_t u - X_H) = 0, \quad \lim_{s \to \pm\infty} u = x/y \right\} /\mathbb{R}$$

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- The count of Floer cylinders with negative punctures defines a $\mathrm{CC}_*(Y)$-DGA-module on $C_+^{-*}(H)\otimes \mathrm{CC}_*(Y)$.
- The count of Floer cylinders with constraint in $\{0\}\times Y$ with negative punctures defines a DGA-module map

$$C_+^{-*}(H)\otimes \mathrm{CC}_*(Y) \to C^{-*+1}(Y)\otimes \mathrm{CC}_*(Y)$$

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Given an exact cobordism W from Y_- to Y_+, by considering holomorphic curves in $\widehat{W}$ with constraints, we get a DGA-module map

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such that the following

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For $Y_+ = \partial(V \times \mathbb{D})$, $\exists x \in C_+^{-*}(H)$ such that $x \otimes 1$ is mapped to $\alpha \otimes 1$, for $\alpha \in \mathrm{Im}(H^*(V \times \mathbb{D}) \to H^*(Y_+))$.

$\Rightarrow \mathrm{CH}(Y_{\Lambda^+}) = 0$, as $\alpha \notin \mathrm{Im}(H^*(W_\Lambda, Y_{\Lambda^+}) \to H^*(Y))$ for the surgery cobordism W_Λ

Producing algebraic (planar) torsions

Spinal open books

- Open book $\mathrm{OB}(V, \phi) = (\partial V \times \mathbb{D}) \cup V_\phi \overset{\mathbb{D}\to\Sigma}{\rightsquigarrow}$ spinal open books (Lisi, Van Horn-Morris, Wendl);
  - spine region: $\partial V \times \Sigma$,
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  - spine region: $\partial V \times \Sigma$,
  - paper region: $\cup V_\phi$.
- $\mathrm{SOB}(V,\phi_1,\ldots,\phi_k)$ is the spinal OB with page V, vertebra S^2 with k disks removed $(= \Sigma_{0,k})$, $\phi_1,\ldots,\phi_k$ are the monodromy, i.e.

$$\mathrm{SOB}(V,\phi_1,\ldots,\phi_k) = (\partial Vt \times \Sigma_{0,k}) \cup_{i=1}^{k} V_{\phi_i}$$

It has a natural map to $\Sigma_{0,k}$.

Theorem (Z. in progress)

Let V be the Brieskorn variety $z_0^{a_0}+\ldots+z_n^{a_n}=1$ for $a_i \gg 0$, then APT *and* AT *(*$\dim \geqslant 7$*) of* $\mathrm{SOB}(V,\phi_1,\ldots,\phi_k)$ *are $k-1$, where ϕ_i are products of negative DS twists with at least one non-trivial.*

Lower bound for torsion

Reeb dynamics on $\mathrm{SOB}(V, \phi_1, \ldots, \phi_k)$:

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Then by virtual dimension computation and homology classes, we have

$$\mathrm{APT}, \mathrm{AT} \geqslant k - 1.$$

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We can get a strong cobordism from $\mathrm{SOB}(V, \phi_1, \ldots, \phi_k)$ to $\mathrm{OB}(V, \phi_{\mathrm{DS}}^{-1})$, which has $k-1$ copies V as symplectic hypersurfaces which make the cobordism non-exact.

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$$\mathrm{APT} \text{ is finite.}$$

Remark

This also produces tight not weakly fillable contact manifolds in dimension $\geqslant 5$.

Upper bound for torsion

To get a precise bound:

- If the vanishing of contact homology for $\mathrm{OB}(V, \phi_{\mathrm{DS}}^{-1})$ comes from holomorphic curves intersecting the binding at most once, since Maurer-Cartan elements have positive intersections with the $k-1$ hypersurfaces, one can conclude that $\mathrm{APT} \leqslant k-1$(This is the case for $(S^{2n+1}, \xi_{\mathrm{ot}})$).

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- $\mathrm{APT} = k-1$ follows from combining the $(+1)$-surgery cobordism and the strong cobordism, i.e. looking at holomorphic curves in the strong cobordism from $\mathrm{SOB}(V, \phi_1, \ldots, \phi_k)$ to $\mathrm{OB}(V, \mathrm{id})$ with a constraint on the Lagrangian disk in the $(+1)$-surgery cobordism.
- Similar arguments apply to AT.

Thank you!