A landscape of contact manifolds via rational SFT

Agustin Moreno
Universität Heidelberg

SFT X, Berlin

J.w. Zhengyi Zhou (Morningside Center, Beijing)

The category of contact manifolds

Goal: Study the *category* of contact manifolds.

The category of contact manifolds

Goal: Study the *category* of contact manifolds.

$\mathfrak{Con}_S$ = category with:

- $\text{Ob}(\mathfrak{Con}_S)$ = contact manifolds.
- $\text{Mor}_{\mathfrak{Con}_S}((M_-, \xi_-), (M_+, \xi_+))$ = strong symplectic cobordisms from M_- to M_+.

Monoidal structure: $(M_1, \xi_1) \otimes (M_2, \xi_2) = (M_1, \xi_1) \sqcup (M_2, \xi_2)$.

The category of contact manifolds

Goal: Study the *category* of contact manifolds.

$\mathfrak{Con}_S$ = category with:

- $\mathrm{Ob}(\mathfrak{Con}_S)$ = contact manifolds.
- $\mathrm{Mor}_{\mathfrak{Con}_S}((M_-, \xi_-), (M_+, \xi_+))$ = strong symplectic cobordisms from M_- to M_+.

Monoidal structure: $(M_1, \xi_1) \otimes (M_2, \xi_2) = (M_1, \xi_1) \sqcup (M_2, \xi_2)$.

$\mathfrak{Con}$ = subcategory with same objects, but morphisms are *exact* symplectic cobordisms.

To study $\mathfrak{Con}$, we will study functors from $\mathfrak{Con}$ to other categories.

The category of contact manifolds

Goal: Study the *category* of contact manifolds.

$\mathfrak{Con}_S$ = category with:

- $\mathrm{Ob}(\mathfrak{Con}_S)$ = contact manifolds.
- $\mathrm{Mor}_{\mathfrak{Con}_S}((M_-, \xi_-), (M_+, \xi_+))$ = strong symplectic cobordisms from M_- to M_+.

Monoidal structure: $(M_1, \xi_1) \otimes (M_2, \xi_2) = (M_1, \xi_1) \sqcup (M_2, \xi_2)$.

$\mathfrak{Con}$ = subcategory with same objects, but morphisms are *exact* symplectic cobordisms.

To study $\mathfrak{Con}$, we will study functors from $\mathfrak{Con}$ to other categories.

- In this talk: cobordisms are *exact*.
- In Zhengyi's talk: cobordisms are *strong*.

Main theorem

Theorem (M. – Zhou '20)

There exists a covariant, monoidal ***complexity/hierarchy*** *functor*

$$\mathrm{H}_{\mathrm{cx}} : \mathfrak{Con} \to \mathcal{H}$$

to a totally ordered category

$$\mathcal{H} = \{\underbrace{0^{\mathrm{APT}} < 1^{\mathrm{APT}} < \ldots < \infty^{\mathrm{APT}}}_{0^{\mathrm{P}}} < \underbrace{0^{\mathrm{SD}} < 1^{\mathrm{SD}} < \ldots < \infty^{\mathrm{SD}}}_{1^{\mathrm{P}}} < 2^{\mathrm{P}} < \ldots < \infty^{\mathrm{P}}\}$$

- **Functoriality**: $M_- \to M_+$ cobordism $\rightsquigarrow \mathrm{H}_{\mathrm{cx}}(M_-) \leqslant \mathrm{H}_{\mathrm{cx}}(M_+)$.
- **Monodial structure on $\mathcal{H}$:**
  - $a^{\mathrm{P}} \otimes 0^{\mathrm{P}} = 0^{\mathrm{P}}$, $a^{\mathrm{P}} \otimes b^{\mathrm{P}} = \max\{a^{\mathrm{P}}, b^{\mathrm{P}}\}$.
  - $a^{\mathrm{APT}} \otimes b^{\mathrm{APT}} = \min\{a^{\mathrm{APT}}, b^{\mathrm{APT}}\}$.
  - $a^{\mathrm{SD}} \otimes b^{\mathrm{SD}} = \max\{a^{\mathrm{SD}}, b^{\mathrm{SD}}\}$.

Intuitively, $\mathsf{H}_{cx}(M, \xi)$ = "*complexity*" of (M, ξ).

Intuitively, $\mathsf{H}_{\mathsf{cx}}(M,\xi)$ = "*complexity*" of (M,ξ).

- P = **planarity**[a]
- APT = **algebraic planar torsion.**
- SD = **order of semi-dilation.**

[a] Ganatra–Siegel define a similar notion $\mathbb{G}\langle p\rangle$ but for *Liouville domains*.

Intuitively, $\mathsf{H}_{\mathsf{cx}}(M, \xi)$ = "*complexity*" of (M, ξ).

- P = **planarity**[a]
- APT = **algebraic planar torsion**.
- SD = **order of semi-dilation**.

[a] Ganatra–Siegel define a similar notion $\mathbb{G}\langle p \rangle$ but for *Liouville domains*.

Main tool: *Rational* (genus zero) symplectic field theory.

- P counts rational curves with a point constraint in symplectizations.
- APT counts rational curves with no negative punctures. Inspired by *algebraic torsion* (Latschev–Wendl).
- SD is defined via the $\mathbb{Q}[u]$-module structure on linearized contact homology (Bourgeois–Oancea).

Relation to contact topology

Theorem (M. – Zhou '20)

The functors above have the following properties.

1. *If Y has planar k-torsion [Wendl], then* $\mathrm{APT}(Y) \leqslant k$.

Relation to contact topology

Theorem (M. – Zhou '20)

The functors above have the following properties.

1. *If Y has planar k-torsion [Wendl], then* $\mathrm{APT}(Y) \leqslant k$.
2. *If Y is overtwisted then* $\mathrm{APT}(Y) = 0$.

Relation to contact topology

Theorem (M. – Zhou '20)

The functors above have the following properties.

1. *If Y has planar k-torsion [Wendl], then* $\mathrm{APT}(Y) \leqslant k$.
2. *If Y is overtwisted then* $\mathrm{APT}(Y) = 0$.
3. *If Y has Giroux torsion [Giroux, MNW], then* $\mathrm{APT}(Y) \leqslant 1$.

Relation to contact topology

Theorem (M. – Zhou '20)

The functors above have the following properties.

1. *If Y has planar k-torsion [Wendl], then* $\mathrm{APT}(Y) \leqslant k$.
2. *If Y is overtwisted then* $\mathrm{APT}(Y) = 0$.
3. *If Y has Giroux torsion [Giroux, MNW], then* $\mathrm{APT}(Y) \leqslant 1$.
4. *If* $\mathrm{APT}(Y) < \infty$*, then Y is not strongly fillable. If Y admits an exact filling then* $\mathrm{P}(Y) \geqslant 1$.

Relation to contact topology

Theorem (M. – Zhou '20)

The functors above have the following properties.

1. *If Y has planar k-torsion [Wendl], then* $\mathrm{APT}(Y) \leqslant k$.
2. *If Y is overtwisted then* $\mathrm{APT}(Y) = 0$.
3. *If Y has Giroux torsion [Giroux, MNW], then* $\mathrm{APT}(Y) \leqslant 1$.
4. *If* $\mathrm{APT}(Y) < \infty$*, then Y is not strongly fillable. If Y admits an exact filling then* $\mathrm{P}(Y) \geqslant 1$.
5. *If Y is an iterated planar open book [Acu] where the initial page has k-punctures, then* $\mathrm{P}(Y) \leqslant k$.

Relation to contact topology

Theorem (M. – Zhou '20)

The functors above have the following properties.

1. *If Y has planar k-torsion [Wendl], then* $\mathrm{APT}(Y) \leqslant k$.
2. *If Y is overtwisted then* $\mathrm{APT}(Y) = 0$.
3. *If Y has Giroux torsion [Giroux, MNW], then* $\mathrm{APT}(Y) \leqslant 1$.
4. *If* $\mathrm{APT}(Y) < \infty$*, then Y is not strongly fillable. If Y admits an exact filling then* $\mathrm{P}(Y) \geqslant 1$.
5. *If Y is an iterated planar open book [Acu] where the initial page has k-punctures, then* $\mathrm{P}(Y) \leqslant k$.
6. *If Y has an exact filling that is not k-uniruled [McLean], then* $\mathrm{P}(Y) \geqslant k + 1$.

Relation to contact topology

Theorem (M. – Zhou '20)

The functors above have the following properties.

1. *If Y has planar k-torsion [Wendl], then* $\mathrm{APT}(Y) \leqslant k$*.*
2. *If Y is overtwisted then* $\mathrm{APT}(Y) = 0$*.*
3. *If Y has Giroux torsion [Giroux, MNW], then* $\mathrm{APT}(Y) \leqslant 1$*.*
4. *If* $\mathrm{APT}(Y) < \infty$*, then Y is not strongly fillable. If Y admits an exact filling then* $\mathrm{P}(Y) \geqslant 1$*.*
5. *If Y is an iterated planar open book [Acu] where the initial page has k-punctures, then* $\mathrm{P}(Y) \leqslant k$*.*
6. *If Y has an exact filling that is not k-uniruled [McLean], then* $\mathrm{P}(Y) \geqslant k+1$*.*
7. $\mathrm{APT}, \mathrm{SD}, \mathrm{P}$ *are all surjective. In particular,* P *is surjective in all fixed odd dimension* $\geqslant 3$*.*

Dynamics: Weinstein conjecture

Theorem (M. – Zhou '20)

If $P(Y) < \infty$, then Y satisfies the Weinstein conjecture.

In other words, counterexamples to the Weinstein conjecture (if any) are *maximally complex*.

Examples

Hyperplane complements

Theorem (M. – Zhou)

Let D_k be k generic hyperplanes in $\mathbb{C}P^n$ for $n \geqslant 2$, then we have the following.

1. $k - 1 \geqslant \mathrm{P}(\partial D_k^c) \geqslant k + 1 - n$ *for* $k > n + 1$.
2. $\mathrm{P}(\partial D_k^c) = k + 1 - n$ *for* $n + 1 < k < \frac{3n-1}{2}$ *and n odd.*
3. $\mathrm{P}(\partial D_k^c) = 2$ *for* $k = n + 1$.
4. $\mathrm{H}_{\mathrm{cx}}(\partial D_k^c) = 0^{\mathrm{SD}}$ *for* $k \leqslant n$.

Hyperplane complements

Theorem (M. – Zhou)

Let D_k be k generic hyperplanes in $\mathbb{C}P^n$ for $n \geqslant 2$, then we have the following.

1. $k-1 \geqslant \mathrm{P}(\partial D_k^c) \geqslant k+1-n$ *for* $k > n+1$.
2. $\mathrm{P}(\partial D_k^c) = k+1-n$ *for* $n+1 < k < \frac{3n-1}{2}$ *and n odd.*
3. $\mathrm{P}(\partial D_k^c) = 2$ *for* $k = n+1$.
4. $\mathrm{H}_{\mathrm{cx}}(\partial D_k^c) = 0^{\mathrm{SD}}$ *for* $k \leqslant n$.

Remarks:

- (1.) obstructs **exact** cobordisms $\partial D^c_{k+r} \to \partial D^c_k$ for $r > n-1, k > n+1$; and (2.) obstructs **exact** cobordisms $\partial D^c_{k+1} \to \partial D^c_k$, for $n+1 < k < \frac{3n-1}{2}$ and n odd.

Hyperplane complements

Theorem (M. – Zhou)

Let D_k be k generic hyperplanes in $\mathbb{C}P^n$ for $n \geqslant 2$, then we have the following.

1. $k - 1 \geqslant \mathrm{P}(\partial D_k^c) \geqslant k + 1 - n$ *for* $k > n + 1$.
2. $\mathrm{P}(\partial D_k^c) = k + 1 - n$ *for* $n + 1 < k < \frac{3n-1}{2}$ *and* n *odd.*
3. $\mathrm{P}(\partial D_k^c) = 2$ *for* $k = n + 1$.
4. $\mathrm{H_{cx}}(\partial D_k^c) = 0^{\mathrm{SD}}$ *for* $k \leqslant n$.

Remarks:

- (1.) obstructs **exact** cobordisms $\partial D_{k+r}^c \to \partial D_k^c$ for $r > n - 1, k > n + 1$; and (2.) obstructs **exact** cobordisms $\partial D_{k+1}^c \to \partial D_k^c$, for $n + 1 < k < \frac{3n-1}{2}$ and n odd.
- But there is always a **strong** cobordism $\partial D_{k+1}^c \to \partial D_k^c$.

Hyperplane complements

Theorem (M. – Zhou)

Let D_k be k generic hyperplanes in $\mathbb{C}P^n$ for $n \geqslant 2$, then we have the following.

1. $k - 1 \geqslant \mathrm{P}(\partial D_k^c) \geqslant k + 1 - n$ *for* $k > n + 1$.
2. $\mathrm{P}(\partial D_k^c) = k + 1 - n$ *for* $n + 1 < k < \frac{3n-1}{2}$ *and n odd.*
3. $\mathrm{P}(\partial D_k^c) = 2$ *for* $k = n + 1$.
4. $\mathrm{H}_{cx}(\partial D_k^c) = 0^{\mathrm{SD}}$ *for* $k \leqslant n$.

Remarks:

- (1.) obstructs **exact** cobordisms $\partial D_{k+r}^c \to \partial D_k^c$ for $r > n - 1, k > n + 1$; and (2.) obstructs **exact** cobordisms $\partial D_{k+1}^c \to \partial D_k^c$, for $n + 1 < k < \frac{3n-1}{2}$ and n odd.
- But there is always a **strong** cobordism $\partial D_{k+1}^c \to \partial D_k^c$.
- And there is always an exact cobordism the other way $\partial D_k^c \to \partial D_{k+1}^c$.

Hyperplane complements

Theorem (M. – Zhou)

Let D_k be k generic hyperplanes in $\mathbb{C}P^n$ for $n \geqslant 2$, then we have the following.

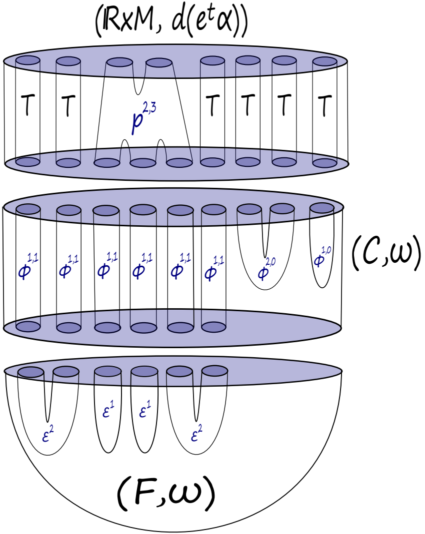
1. $k - 1 \geqslant \mathrm{P}(\partial D_k^c) \geqslant k + 1 - n$ *for* $k > n + 1$.
2. $\mathrm{P}(\partial D_k^c) = k + 1 - n$ *for* $n + 1 < k < \frac{3n-1}{2}$ *and n odd.*
3. $\mathrm{P}(\partial D_k^c) = 2$ *for* $k = n + 1$.
4. $\mathrm{H}_{cx}(\partial D_k^c) = 0^{\mathrm{SD}}$ *for* $k \leqslant n$.

Remarks:

- (1.) obstructs **exact** cobordisms $\partial D_{k+r}^c \to \partial D_k^c$ for $r > n - 1, k > n + 1$; and (2.) obstructs **exact** cobordisms $\partial D_{k+1}^c \to \partial D_k^c$, for $n + 1 < k < \frac{3n-1}{2}$ and n odd.
- But there is always a **strong** cobordism $\partial D_{k+1}^c \to \partial D_k^c$.
- And there is always an exact cobordism the other way $\partial D_k^c \to \partial D_{k+1}^c$.
- If $k \leqslant n$, all ∂D_k^c have exact cobordisms both ways.

Algebraic aspects

Rational holomorphic buildings



We wish to capture the combinatorics of boundary degenerations of rational curves into rational holomorphic buildings.

Conventions

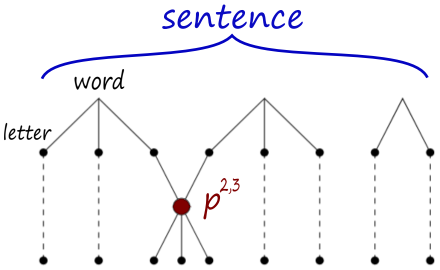
V = $\mathbb{Z}_2$-graded **k**-v.space. Let $S^j V = V^{\otimes j}/S_j$, S_j permutation group.

Notation:

- $SV = \bigoplus_{j \geqslant 0} S^j V$ symmetric algebra (words).
- $\overline{S}V = \bigoplus_{j \geqslant 1} S^j V$ non-unital symmetric algebra (non-empty words).
- $\overline{B}^k V = \bigoplus_{j=1}^{k} S^j V$ (non-empty words with at most k letters).
- $EV = \overline{S}SV$ (non-empty sentences).
- $E^k V = \bigoplus_{j=1}^{k} S^j SV \subset EV$ (sentences with at most k words).

BL_∞ algebras

Given linear operators $p^{k,l} : S^kV \to S^lV$ for $k \geqslant 1, l \geqslant 0$, we can define a map $\widehat{p} : EV \to EV$, most easily described by trees.

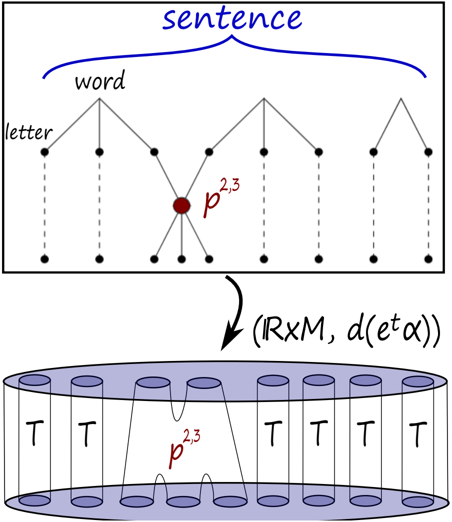


$\widehat{p}$ is obtained by summing over all glued trees.

Definition

$(V, \widehat{p})$ is a BL_∞−**algebra** if $|\widehat{p}| = 1$ and $\widehat{p}^2 = 0$.

Rational curves



Graphs represent counts of rational holomorphic curves in symplectizations.

Remarks

- BL_∞-algebras are a genus zero specialization of IBL_∞-algebras (Cieliebak–Fukaya–Latschev).

Remarks

- BL_∞-algebras are a genus zero specialization of IBL_∞-algebras (Cieliebak–Fukaya–Latschev).
- Assuming $p^{k,0} = 0$ for all k, then $p^{1,1}$ differential, and on its *homology*:
  1. $p^{2,1}$ Lie bracket;
  2. $p^{1,2}$ co-Lie bracket;
  3. $p^{1,2} \circ p^{2,1} = 0$.

  I.e. homology of $p^{1,1}$ is a **bi-Lie algebra**.

Remarks

- BL_∞-algebras are a genus zero specialization of IBL_∞-algebras (Cieliebak–Fukaya–Latschev).
- Assuming $p^{k,0} = 0$ for all k, then $p^{1,1}$ differential, and on its *homology*:
  1. $p^{2,1}$ Lie bracket;
  2. $p^{1,2}$ co-Lie bracket;
  3. $p^{1,2} \circ p^{2,1} = 0$.

  I.e. homology of $p^{1,1}$ is a **bi-Lie algebra**.
- Under same assumption, $p^{k,1}$ induces an L_∞ algebra on V (Siegel).

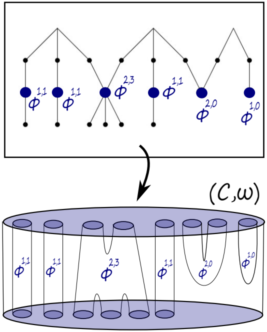
Remarks

- BL_∞-algebras are a genus zero specialization of IBL_∞-algebras (Cieliebak–Fukaya–Latschev).
- Assuming $p^{k,0} = 0$ for all k, then $p^{1,1}$ differential, and on its *homology*:
  1. $p^{2,1}$ Lie bracket;
  2. $p^{1,2}$ co-Lie bracket;
  3. $p^{1,2} \circ p^{2,1} = 0$.

  I.e. homology of $p^{1,1}$ is a **bi-Lie algebra**.
- Under same assumption, $p^{k,1}$ induces an L_∞ algebra on V (Siegel).
- In general, a BL_∞ algebra $(V, \widehat{p})$ induces an L_∞ algebra $(SV, \widehat{\ell})$.

Morphisms

Given linear maps $\{\phi^{k,l} : S^k V \to S^l V'\}_{k\geqslant 1, l\geqslant 0}$, we construct $\widehat{\phi} : EV \to EV'$ similarly as before.



Morphism graphs represent counts of rational holomorphic curves in symplectic cobordisms.

Definition

$\widehat{\phi}$ is a BL_∞ **morphism** from (V, p) to (V', p') if $\widehat{\phi} \circ \widehat{p} = \widehat{p}' \circ \widehat{\phi}$ and $|\widehat{\phi}| = 0$.

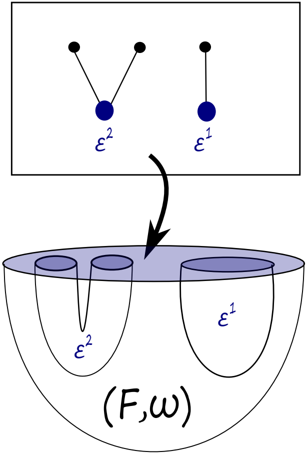
Augmentations

The zero BL_∞ algebra $\mathbf{0}$ has $S^k\mathbf{0} = \mathbf{k}$ for all k and $\widehat{p} = 0$.

Definition

A BL_∞ **augmentation** is a BL_∞ morphism $\epsilon : (V, \widehat{p}) \to \mathbf{0}$, i.e. a family of maps $\epsilon^k : S^k V \to \mathbf{k}$ so that $|\widehat{\epsilon}| = 0$ and $\widehat{\epsilon} \circ \widehat{p} = 0$.

Augmentations



An augmentation algebraically represents counts of rational holomorphic curves with negative ends on a symplectic filling. It still makes sense algebraically if there is no filling.

Torsion

Given $(V, \widehat{p})$ BL_∞-algebra, the unit

$$1_V \in H_*(E^k V)$$

is the image of the unit

$$1_{\mathbf{0}} \in \mathbf{k} = E^1\mathbf{0} \subset H_*(E^k\mathbf{0}) = E^K\mathbf{0}$$

under the map $H_*(E^k\mathbf{0}) \to H_*(E^k V)$ induced by $\mathbf{0} \to (V, \widehat{p})$.

Torsion

Given $(V, \widehat{p})$ BL_∞-algebra, the unit

$$1_V \in H_*(E^k V)$$

is the image of the unit

$$1_{\mathbf{0}} \in \mathbf{k} = E^1\mathbf{0} \subset H_*(E^k\mathbf{0}) = E^K\mathbf{0}$$

under the map $H_*(E^k\mathbf{0}) \to H_*(E^k V)$ induced by $\mathbf{0} \to (V, \widehat{p})$.

Definition

The **torsion** of a BL_∞ algebra V is

$$T(V) := \min\{k-1 | 1_V = 0 \in H^*(E^k V), k \geqslant 1\}.$$

Here the minimum of an empty set is defined to be ∞.

This is the algebraic counterpart of APT: roughly speaking, it gives the smallest number of positive punctures (minus one) of curves killing the unit in homology.

Functoriality of torsion

BL_∞ morphisms preserve the sentence length filtration

$$E^1V \subset E^2V \subset \dots$$

Then:

If $(V, \widehat{p}) \to (V', \widehat{p'})$ BL_∞ morphism, we have

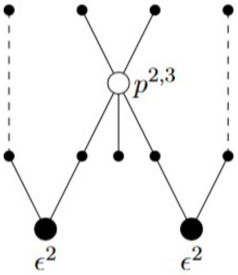
$$T(V) \geqslant T(V').$$

In particular, if $(V, \widehat{p})$ has an augmentation, then $T(V) = \infty$.

Linearized theory

Given $(V, \widehat{p})$ BL_∞-algebra and $\epsilon : (V, \widehat{p}) \to \mathbf{0}$ augmentation,

$\rightsquigarrow (V, \widehat{p}_\epsilon)$ linearized BL_∞ – algebra, with $p_\epsilon^{k,0} = 0$.



A $p_\epsilon^{4,1}$ component. In general, to obtain $p_\epsilon^{k,l}$, we sum over connected trees with *exactly* one $p^{k',l'}$, and several ϵ^j.

Pointed maps and linearizations

Given $p_\bullet^{k,l} : S^k V \to S^l V, k \geqslant 1, l \geqslant 0$ linear maps, we similarly define $\widehat{p}_\bullet : EV \to EV$, but now $|\widehat{p}_\bullet| \neq 1$ in general.

Definition

$(V, \{p_\bullet^{k,l}\})$ is a **pointed map** for $(V, \widehat{p})$ if $\widehat{p}_\bullet \circ \widehat{p} = (-1)^{|\widehat{p}_\bullet|} \widehat{p} \circ \widehat{p}_\bullet$.

Pointed maps and linearizations

Given $p_\bullet^{k,l} : S^k V \to S^l V, k \geqslant 1, l \geqslant 0$ linear maps, we similarly define $\widehat{p}_\bullet : EV \to EV$, but now $|\widehat{p}_\bullet| \neq 1$ in general.

Definition

$(V, \{p_\bullet^{k,l}\})$ is a **pointed map** for $(V, \widehat{p})$ if $\widehat{p}_\bullet \circ \widehat{p} = (-1)^{|\widehat{p}_\bullet|}\widehat{p} \circ \widehat{p}_\bullet$.

Pointed maps represent counts of rational holomorphic curves with one interior marked point constrained on a cycle $Z \in H_*(Y)$.

$\rightsquigarrow |\widehat{p}_\bullet| = \deg(Z)$ ($= 0$ if $Z = \mathrm{pt}$).

Pointed maps and linearizations

Given $p_\bullet^{k,l} : S^k V \to S^l V$, $k \geqslant 1, l \geqslant 0$ linear maps, we similarly define $\widehat{p}_\bullet : EV \to EV$, but now $|\widehat{p}_\bullet| \neq 1$ in general.

Definition

$(V, \{p_\bullet^{k,l}\})$ is a **pointed map** for $(V, \widehat{p})$ if $\widehat{p}_\bullet \circ \widehat{p} = (-1)^{|\widehat{p}_\bullet|} \widehat{p} \circ \widehat{p}_\bullet$.

Pointed maps represent counts of rational holomorphic curves with one interior marked point constrained on a cycle $Z \in H_*(Y)$.

$\rightsquigarrow |\widehat{p}_\bullet| = \deg(Z)$ ($= 0$ if $Z = \mathrm{pt}$).

Given a linearization $\epsilon : (V, \widehat{p}) \to \mathbf{0}$,

$\widehat{p}_\bullet \rightsquigarrow \widehat{p}_{\bullet,\epsilon}$ linearized pointed map for $(V, \widehat{p}_\epsilon)$. But $p_{\bullet,\epsilon}^{k,0} \neq 0$ in general.

Order

Let $\ell_\epsilon^k = p_\epsilon^{k,1} \rightsquigarrow \widehat{\ell}_\epsilon$ (an L_∞ structure on $V[-1]$). We get a chain map

$$\widehat{\ell}_{\bullet,\epsilon} = \sum_{k \geqslant 1} p_{\bullet,\epsilon}^{k,0} : (\overline{S}V, \widehat{\ell}_\epsilon) \to \mathbf{k}.$$

Definition

The $(\epsilon, \widehat{p}_\bullet)$-**order** of $(V, \widehat{p})$ is

$$O(V, \epsilon, \widehat{p}_\bullet) := \min \left\{ k : 1 \in \operatorname{Im} \widehat{\ell}_{\bullet,\epsilon}|_{H_*(\overline{B}^k V, \widehat{\ell}_\epsilon)} \right\},$$

where the minimum of an empty set is defined to be ∞.

This is the algebraic counterpart to planarity: roughly speaking, it gives the smallest number of positive punctures of a point-constrained curve hitting the unit in **k**.

Functoriality of order

Given:

- BL_∞ morphism $\widehat{\phi} : (V, \widehat{p}) \to (V', \widehat{q})$;
- maps $\phi_\bullet^{k,l} : S^k V \to S^l V'$.

$\rightsquigarrow \widehat{\phi}_\bullet : EV \to EV'$ via graphs.

- $\widehat{p}_\bullet, \widehat{q}_\bullet$ are two pointed maps for $(V, \widehat{p}), (V', \widehat{q})$ respectively, of the same degree.

Definition

We say $\widehat{p}_\bullet, \widehat{q}_\bullet, \widehat{\phi}$ are **compatible**, if there are $\phi_\bullet^{k,l}$ such that

$$\widehat{q}_\bullet \circ \widehat{\phi} - (-1)^{|\widehat{q}_\bullet|}\widehat{\phi} \circ \widehat{p}_\bullet = \widehat{q} \circ \widehat{\phi}_\bullet - (-1)^{|\widehat{\phi}_\bullet|}\widehat{\phi}_\bullet \circ \widehat{p}$$

and $|\widehat{\phi}_\bullet| = |\widehat{p}_\bullet| + 1$.

Functoriality of order

Assume $\widehat{p}_\bullet, \widehat{q}_\bullet, \widehat{\phi}$ are compatible and $|\widehat{p}_\bullet| = |\widehat{q}_\bullet| = 0$. Then:

For any BL_∞ augmentation ϵ of V', we have

$$O(V, \epsilon \circ \widehat{\phi}, \widehat{p}_\bullet) \geqslant O(V', \epsilon, \widehat{q}_\bullet).$$

Geometric aspects

RSFT as a BL_∞-algebra

(Y^{2n-1}, α) strict contact manifold. $\Lambda =$ Novikov field.

- $V_\alpha =$ free Λ-module generated by (good) Reeb orbits.

RSFT as a BL_∞-algebra

(Y^{2n-1}, α) strict contact manifold. $\Lambda =$ Novikov field.

- $V_\alpha =$ free Λ-module generated by (good) Reeb orbits.
- $CHA(Y) = SV_\alpha$ contact homology algebra.

RSFT as a BL_∞-algebra

(Y^{2n-1}, α) strict contact manifold. $\Lambda =$ Novikov field.

- $V_\alpha =$ free Λ-module generated by (good) Reeb orbits.
- $CHA(Y) = SV_\alpha$ contact homology algebra.
- γ orbit $\rightsquigarrow$ generator $q_\gamma, |q_\gamma| = \mu_{CZ}(\gamma) + n - 3$ mod 2.

RSFT as a BL_∞-algebra

(Y^{2n-1}, α) strict contact manifold. $\Lambda =$ Novikov field.

- $V_\alpha =$ free Λ-module generated by (good) Reeb orbits.
- $CHA(Y) = SV_\alpha$ contact homology algebra.
- γ orbit $\rightsquigarrow$ generator $q_\gamma, |q_\gamma| = \mu_{CZ}(\gamma) + n - 3 \bmod 2$.
- $\Gamma = \{\underbrace{\eta_1, \ldots, \eta_1}_{i_1}, \ldots, \underbrace{\eta_m, \ldots, \eta_m}_{i_m}\}$ ordered orbit set, $\eta_i \neq \eta_j$, $i \neq j$, $\sum i_j = l =: |\Gamma|$.

RSFT as a BL_∞-algebra

(Y^{2n-1}, α) strict contact manifold. $\Lambda =$ Novikov field.

- $V_\alpha =$ free Λ-module generated by (good) Reeb orbits.
- $CHA(Y) = SV_\alpha$ contact homology algebra.
- γ orbit $\rightsquigarrow$ generator $q_\gamma, |q_\gamma| = \mu_{CZ}(\gamma) + n - 3 \bmod 2$.
- $\Gamma = \{\underbrace{\eta_1, \ldots, \eta_1}_{i_1}, \ldots, \underbrace{\eta_m, \ldots, \eta_m}_{i_m}\}$ ordered orbit set, $\eta_i \neq \eta_j$, $i \neq j$, $\sum i_j = l =: |\Gamma|$.
- $\mu_\Gamma = i_1! \ldots i_m!$, $\kappa_\Gamma = \kappa_{\eta_1}^{i_1} \ldots \kappa_{\eta_m}^{i_m}$ product of multiplicities, and $q^\Gamma = q_{\eta_1}^{i_1} \ldots q_{\eta_m}^{i_m}$.

RSFT as a BL_∞-algebra

$\overline{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-)$ = compactified moduli space of rational curves in $\mathbb{R} \times Y$ asymptotic to $\Gamma^\pm$ in homology class $A \in H_2(Y, \Gamma^- \cup \Gamma^+)$.

$$p^{k,l}(q^{\Gamma^+}) = \sum_{|\Gamma^-|=l} \#\overline{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-) \frac{T^{\int_A d\alpha}}{\mu_{\Gamma^+}\mu_{\Gamma^-}\kappa_{\Gamma^-}} q^{\Gamma^-}.$$

RSFT as a BL_∞-algebra

$\overline{\mathcal{M}}_{Y,A}(\Gamma^+,\Gamma^-)$ = compactified moduli space of rational curves in $\mathbb{R}\times Y$ asymptotic to $\Gamma^\pm$ in homology class $A \in H_2(Y,\Gamma^-\cup\Gamma^+)$.

$$p^{k,l}(q^{\Gamma^+}) = \sum_{|\Gamma^-|=l} \#\overline{\mathcal{M}}_{Y,A}(\Gamma^+,\Gamma^-)\frac{T^{\int_A d\alpha}}{\mu_{\Gamma^+}\mu_{\Gamma^-}\kappa_{\Gamma^-}}q^{\Gamma^-}.$$

Fix a point o in Y, $\overline{\mathcal{M}}_{Y,A,o}(\Gamma^+,\Gamma^-)$ = rational curves passing through o.

$$p_\bullet^{k,l}(q^{\Gamma^+}) = \sum_{|\Gamma^-|=l} \#\overline{\mathcal{M}}_{Y,A,o}(\Gamma^+,\Gamma^-)\frac{T^{\int_A d\alpha}}{\mu_{\Gamma^+}\mu_{\Gamma^-}\kappa_{\Gamma^-}}q^{\Gamma^-}.$$

RSFT as a BL_∞-algebra

$(X, \omega = d\lambda)$ exact cobordism from (Y_-, α_-) to (Y_+, α_+).

$$\phi^{k,l}(q^{\Gamma^+}) = \sum_{|\Gamma^-|=l} \#\overline{\mathcal{M}}_{X,A}(\Gamma^+, \Gamma^-) \frac{T^{\int_A \omega}}{\mu_{\Gamma^+}\mu_{\Gamma^-}\kappa_{\Gamma^-}} q^{\Gamma^-}.$$

If $Y_- = \emptyset$, i.e. X filling, define $\epsilon^k = \phi^{k,0}$ as linearization.

RSFT as a BL_∞-algebra

$(X, \omega = d\lambda)$ exact cobordism from (Y_-, α_-) to (Y_+, α_+).

$$\phi^{k,l}(q^{\Gamma^+}) = \sum_{|\Gamma^-|=l} \#\overline{\mathcal{M}}_{X,A}(\Gamma^+, \Gamma^-) \frac{T^{\int_A \omega}}{\mu_{\Gamma^+}\mu_{\Gamma^-}\kappa_{\Gamma^-}} q^{\Gamma^-}.$$

If $Y_- = \emptyset$, i.e. X filling, define $\epsilon^k = \phi^{k,0}$ as linearization.

Fix points $o_\pm$ in $Y_\pm$, and a curve γ in X joining them, $\overline{\mathcal{M}}_{X,A,\gamma}(\Gamma^+, \Gamma^-)$ = moduli of curves in X passing through γ.

$$\phi_\bullet^{k,l}(q^{\Gamma^+}) = \sum_{|\Gamma^-|=l} \#\overline{\mathcal{M}}_{X,A,\gamma}(\Gamma^+, \Gamma^-) \frac{T^{\int_A \omega}}{\mu_{\Gamma^+}\mu_{\Gamma^-}\kappa_{\Gamma^-}} q^{\Gamma^-}.$$

APT and P

Definition

- The **algebraic planar torsion** of (Y, ξ) is the torsion

$$\mathrm{APT}(Y, \xi) = T(V_\alpha, \widehat{p}).$$

This is independent of α.

APT and P

Definition

- The **algebraic planar torsion** of (Y, ξ) is the torsion

$$\mathrm{APT}(Y, \xi) = T(V_\alpha, \widehat{p}).$$

  This is independent of α.

- The **planarity** of (Y, ξ) is the maximal order over all augmentations,

$$P(Y) := \max_{\epsilon \in \mathrm{Aug}_{\mathbb{Q}}(V_\alpha)} \left\{ O(V_\alpha, \epsilon, \widehat{p}_\bullet) \right\},$$

  where the maximum of an empty set is defined to be zero. This is independent of α.

linearized contact homology

W filling of (Y,ξ), $\epsilon = \epsilon_W$ induced augmentation on $CHA(Y)$.

$$\rightsquigarrow H_*(\overline{B}^1 V_\alpha, \widehat{\ell}_\epsilon) = H_*(V_\alpha, \ell^1_\epsilon) \cong LCH_*(W) \cong SH^{2n-3-*}_{+,S^1}(W)$$

linearized contact homology, or positive S^1-equivariant symplectic cohomology (Bourgeois–Oancea).

linearized contact homology

W filling of (Y, ξ), $\epsilon = \epsilon_W$ induced augmentation on $CHA(Y)$.

$$\rightsquigarrow H_*(\overline{B}^1 V_\alpha, \widehat{\ell}_\epsilon) = H_*(V_\alpha, \ell^1_\epsilon) \cong LCH_*(W) \cong SH^{2n-3-*}_{+,S^1}(W)$$

linearized contact homology, or positive S^1-equivariant symplectic cohomology (Bourgeois–Oancea). These are $\mathbb{Q}[U]$-modules, and

- U has degree 2 on $H_*(V_\alpha, \ell^1_\epsilon)$;
- for each x, there exists k such that $U^k(x) = 0$.

Moreover, the U-map makes sense for *arbitrary* augmentations on $H_*(V_\alpha, \ell^1_\epsilon)$.

SD

Let $(Y, \xi = \ker \alpha)$ with $P(Y) = 1$, $o \in Y$, ϵ an augmentation, and $\ell^1_{\bullet,\epsilon}$ associated pointed map.

Definition

The (ϵ, o)-**order of semi-dilation** of (Y, ξ) is

$$\mathrm{SD}(Y, \xi, \epsilon, o) =$$

$$\min\left\{k : \text{ there exists } x \in H_*(V_\alpha, \ell^1_\epsilon) \text{ with } U^{k+1}(x) = 0, \ell^1_{\bullet,\epsilon}(x) = 1\right\}.$$

The **order of semi-dilation** of (Y, ξ) is

$$\mathrm{SD}(Y, \xi) = \max\left\{\mathrm{SD}(Y, \xi, \epsilon, o) : \epsilon \in \mathrm{Aug}_{\mathbb{Q}}(V_\alpha), o \in Y\right\}.$$

Only depends on the contact structure, and is functorial.

Summary

- RSFT can be defined as a BL_∞-algebra (cf. Eliashberg–Givental–Hofer formalism).

Summary

- RSFT can be defined as a BL_∞-algebra (cf. Eliashberg–Givental–Hofer formalism).
- This notion captures: differentials, morphisms, and augmentations, and their pointed versions.

Summary

- RSFT can be defined as a BL_∞-algebra (cf. Eliashberg–Givental–Hofer formalism).
- This notion captures: differentials, morphisms, and augmentations, and their pointed versions.
- Can define two invariants algebraically (torsion, order).

Summary

- RSFT can be defined as a BL_∞-algebra (cf. Eliashberg–Givental–Hofer formalism).
- This notion captures: differentials, morphisms, and augmentations, and their pointed versions.
- Can define two invariants algebraically (torsion, order).
- These translate to three geometric invariants (APT, P, SD).

Summary

- RSFT can be defined as a BL_∞-algebra (cf. Eliashberg–Givental–Hofer formalism).
- This notion captures: differentials, morphisms, and augmentations, and their pointed versions.
- Can define two invariants algebraically (torsion, order).
- These translate to three geometric invariants (APT, P, SD).
- These are functorial, and independent on contact form. The invariants are not expected to depend on virtual machinery: functoriality and our examples are independent of it under minimal requirements (i.e. geometric count = virtual count when transversality holds).

Summary

- RSFT can be defined as a BL_∞-algebra (cf. Eliashberg–Givental–Hofer formalism).
- This notion captures: differentials, morphisms, and augmentations, and their pointed versions.
- Can define two invariants algebraically (torsion, order).
- These translate to three geometric invariants (APT, P, SD).
- These are functorial, and independent on contact form. The invariants are not expected to depend on virtual machinery: functoriality and our examples are independent of it under minimal requirements (i.e. geometric count = virtual count when transversality holds).
- APT and P have been completely implemented and rigorosuly defined. SD can be implemented with current technology (work in progress; relies on implementing linearized contact homology in Pardon's VFC).

Summary

- RSFT can be defined as a BL_∞-algebra (cf. Eliashberg–Givental–Hofer formalism).
- This notion captures: differentials, morphisms, and augmentations, and their pointed versions.
- Can define two invariants algebraically (torsion, order).
- These translate to three geometric invariants (APT, P, SD).
- These are functorial, and independent on contact form. The invariants are not expected to depend on virtual machinery: functoriality and our examples are independent of it under minimal requirements (i.e. geometric count = virtual count when transversality holds).
- APT and P have been completely implemented and rigorosuly defined. SD can be implemented with current technology (work in progress; relies on implementing linearized contact homology in Pardon's VFC).
- invariants can be used to obstruct existence of cobordisms and measure complexity of contact manifolds.

Summary

- RSFT can be defined as a BL_∞-algebra (cf. Eliashberg–Givental–Hofer formalism).
- This notion captures: differentials, morphisms, and augmentations, and their pointed versions.
- Can define two invariants algebraically (torsion, order).
- These translate to three geometric invariants (APT, P, SD).
- These are functorial, and independent on contact form. The invariants are not expected to depend on virtual machinery: functoriality and our examples are independent of it under minimal requirements (i.e. geometric count = virtual count when transversality holds).
- APT and P have been completely implemented and rigorosuly defined. SD can be implemented with current technology (work in progress; relies on implementing linearized contact homology in Pardon's VFC).
- invariants can be used to obstruct existence of cobordisms and measure complexity of contact manifolds.
- Invariants can be computed or estimated explicitly.

Thank you!