Symplectic geometry of Anosov flows in dimension three

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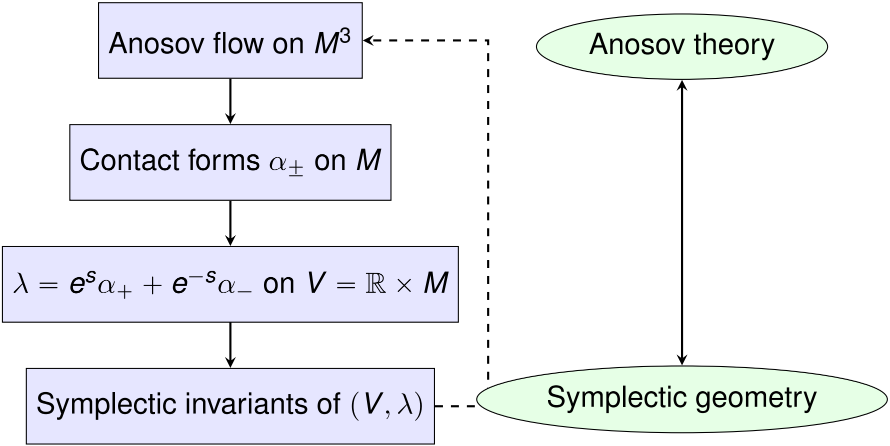
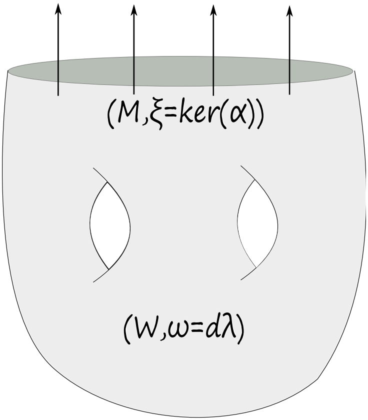
Objective: illustrate the symplectic geometry of Anosov flows in dimension three.



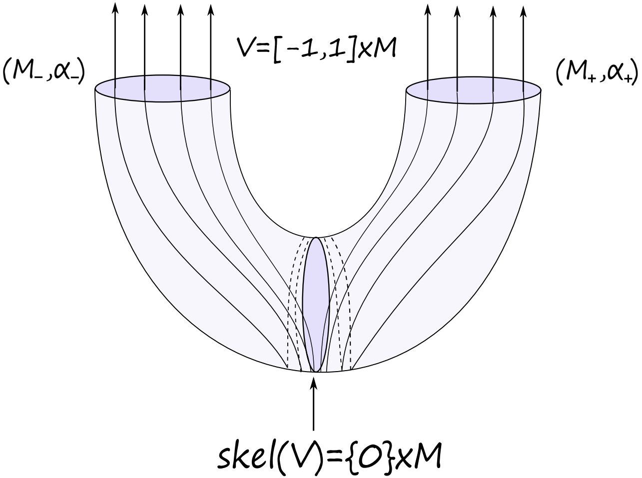
Liouville domains

A **Liouville domain** is $(W, \omega = d\lambda)$ with contact boundary $(M = \partial W, \xi = \ker \alpha)$.



A Liouville domain.

Anosov-Liouville domains



An **Anosov-Liouville** domain $V = [-1, 1] \times M$ contracts to the skeleton $\mathrm{skel}(V) = \{0\} \times M$ under the negative Liouville flow, which is Anosov on $\mathrm{skel}(V)$ (*algebraic* case). Note V is **non-Weinstein**.

Definition

An **Anosov-Liouville structure** on $\mathbb{R}_s \times M^3$ is a smooth Liouville form λ of the form

$$\lambda = e^{-s}\alpha_- + e^s\alpha_+ \quad (1)$$

where (α_-, α_+) are contact forms satisfying:

1. $\xi_\pm = \ker \alpha_\pm$ are transversal.
2. The 1-distribution $\xi_- \cap \xi_+$ is generated by an Anosov vector field X.
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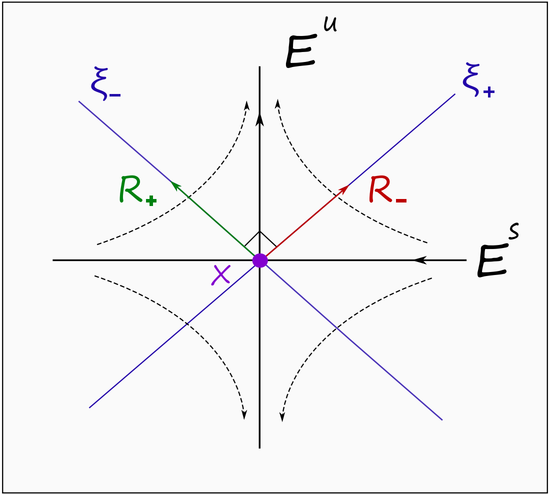
The vector field X (or its flow ϕ_t) is **supported** by the Anosov-Liouville structure. The pair $(\xi_+ = \ker \alpha_+, \xi_- = \ker \alpha_-)$ is a **bi-contact structure**, and (α_-, α_+) is a **Liouville pair**.

Bi-contact structure



The bi-contact structure $\xi_+ \cap \xi_- = \langle X \rangle$. In the algebraic cases, $R_\pm \in \xi_\mp$. In general, both transverse to E^u, E^s, then $\alpha_\pm$ **hypertight** by tautness of $\mathcal{F}^u, \mathcal{F}^s$.

Symplectic invariants of an Anosov flow

Theorem (Massoni '22)

Given $\phi_t : M \to M$ a C^∞ Anosov flow on a closed 3-fold, the space of Anosov-Liouville structures on $V = \mathbb{R} \times M$ that support ϕ_t is non-empty and contractible. The map

$$\{\textit{Anosov-Liouville structures}\} \to \{\textit{Anosov flows}\} \Big/ \textit{reparam.}$$

$$\lambda = e^s \alpha_+ + e^{-s} \alpha_- \mapsto \xi_+ \cap \xi_-$$

is a fibration with contractible fibers, then it is a homotopy equivalence.

The topology is the C^∞-topology (cf. structural stability).

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Corollary (Massoni '22)

If λ supports ϕ_t, every symplectic invariant of (V, λ) is an ***invariant of the flow****, up to homotopies in the space of Anosov flows.*

Symplectic invariants

Some **examples** of symplectic invariants:

1. Symplectic cohomology of (V, λ).
2. Rabinowitz–Floer cohomology of (V, λ).
3. The wrapped Fukaya category $\mathcal{W}(V)$.

$\vdots$

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Plus several algebraic structures (products, open-closed map,...).

With Kai Cieliebak, Oleg Lazarev and Thomas Massoni [CLMM] we calculated invariants of the classical algebraic cases.

The orbit category: Lagrangians on V

Given X Anosov supported by $\lambda = e^s\alpha_+ + e^{-s}\alpha_-$, $\langle X \rangle = \xi_+ \cap \xi_-$,

$\Rightarrow$ Every closed orbit Λ is **Legendrian** for $\xi_\pm$.

$$\mathcal{L}_\Lambda := \mathbb{R} \times \Lambda \Rightarrow \lambda|_{\mathcal{L}_\Lambda} \equiv 0 \Rightarrow \omega|_{\mathcal{L}_\Lambda} = d\lambda|_{\mathcal{L}_\Lambda} \equiv 0$$

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In [CLMM] we studied the A_∞ *sub-category* of $\mathcal{W}(V)$, generated by the orbits of the flow, the **orbit category** $\mathcal{W}_0(V)$.

Open-closed map

We have $\mathcal{L}_\Lambda$ is contained in the unstable manifold of Λ. By analogy to the Weinstein case, we can ask:

Q: Does the family $\{\mathcal{L}_\Lambda\}$ (split-)generate $\mathcal{W}(V)$?

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First try: Abouzaid's generation criterion. *However:*

Theorem (Cieliebak–Lazarev–Massoni–M. '22)

*The open-closed map $\mathcal{OC}_0 : HH_{*-2}(\mathcal{W}_0(V)) \to SH^*(V)$ does **not** hit the unit. Moreover:*

- *Any two Lagrangians $\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}$ with $\Lambda \neq \Lambda'$ are **not** quasi-isomorphic in $\mathcal{W}(V)$,*
- *$\mathcal{W}_0(V)$ is **not** split-generated by finitely many objects $\mathcal{L}_\Lambda$,*
- *$\mathcal{W}_0(V)$ is **not** homologically smooth.*

Dichotomy

Corollary

We have the two possibilities:

- *The family* $\{\mathcal{L}_\Lambda\}$ *split-generate, in which case* $\mathcal{W}(V)$ *is not homologically smooth; or*
- *There exist "mystery" Lagrangians.*

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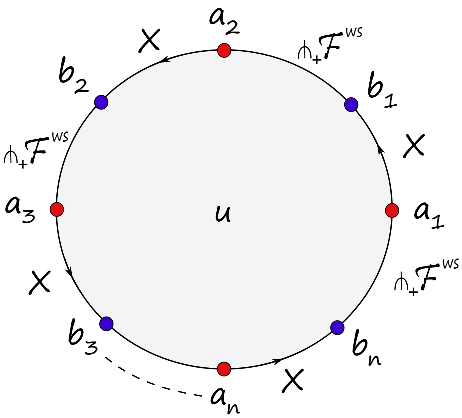
Remark:

- In all examples we considered, we do not know how to construct Lagrangians which are not expected to lie in the split closure of $\mathcal{W}_0(V)$.
- The first option would be in stark contrast to the Weinstein case.

Ingredients for proof



The key dynamical input is that the weak stable/unstable foliations are *taut*, and hence admit no contractible transverse loops. A disk as above can be perturbed to induce such loop.

Ingredients for proof

This precludes most contributions to $\mathcal{OC}_0 : HH_{*-2}(\mathcal{W}_0(V)) \to SH^*(V)$, which splits into *contractible* and *non-contractible* parts,

$$\mathcal{OC}_0 = \mathcal{OC}_0^c \oplus \mathcal{OC}_0^{nc}.$$

Here,

$$\mathcal{OC}_0^c : HH_{*-2}^c \to SH_c^*(V) \cong H^*(M),$$
$$\mathcal{OC}_0^{nc} : HH_{*-2}^{nc} \to SH_{nc}^*(V).$$

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Moreover, there is an isomorphism

$$HH^c_* \cong \bigoplus_{\Lambda} HH_*(C^*(S^1)).$$

This is supported in degrees $0, 1$ then

$$\mathrm{Im}(\mathcal{OC}_0^c) \subseteq H^2(M;\mathbb{Z}) \oplus H^3(M;\mathbb{Z}) \text{ does not contain the unit.}$$

Ingredients for proof

$\mathcal{C}$ collection of orbits, $\Lambda \notin \mathcal{C}$, $\mathcal{C}' = \mathcal{A} \cup \{\Lambda\}$. Let $\mathcal{A}, \mathcal{A}'$ full subcategories generated by $\mathcal{C}, \mathcal{C}'$.

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$$\iota_* : HH_*(\mathcal{A}) \to HH_*(\mathcal{A}')$$

splits as a sum

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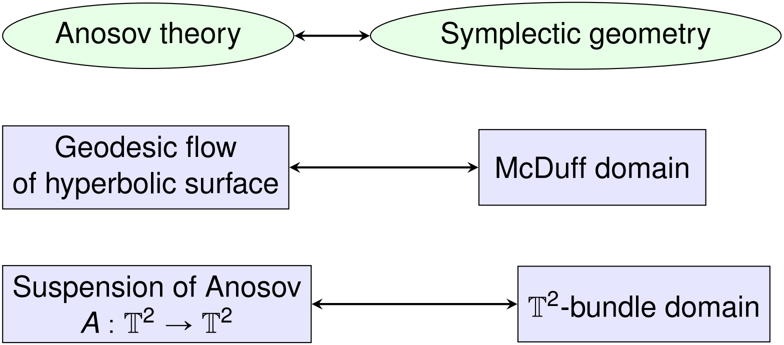
$\Rightarrow \iota_*$ **not** an isomorphism.

$\Rightarrow \Lambda$ **not** generated by $\mathcal{C}$.

Basic examples: algebraic flows



McDuff domains

(Σ, g) hyperbolic surface, $\sigma = vol_g \in \Omega^2(\Sigma)$. The **magnetic** cotangent bundle is

$$(T^*\Sigma, \omega_\sigma = d\lambda_{\text{std}} + \pi^*\sigma),$$

with $\pi : T^*\Sigma \to \Sigma$ natural projection.

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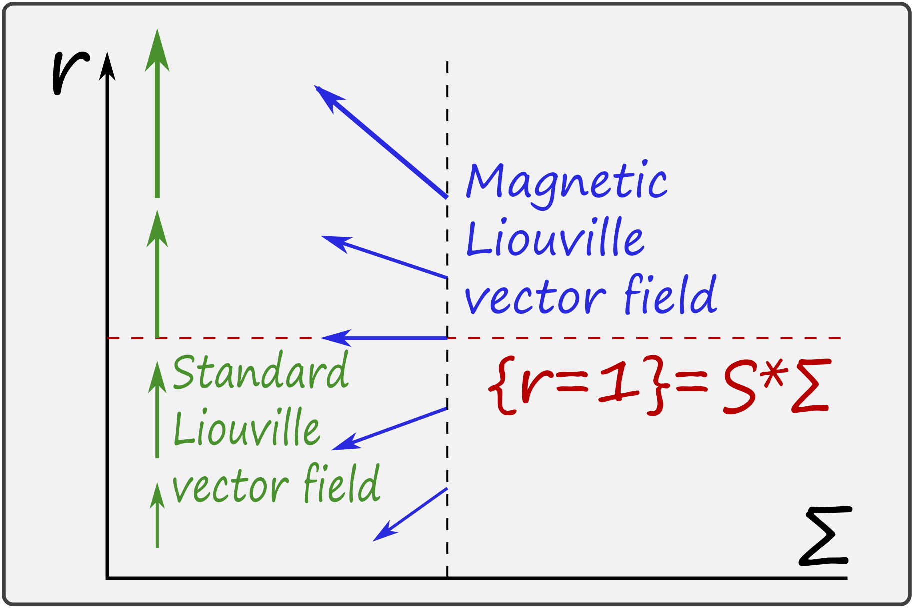
$$(\xi_+, \xi_-) = (\ker \alpha_+, \ker \alpha_-)$$

bi-contact structure, supporting the **conormal** geodesic flow. Lagrangian cylinders are $\mathcal{L}_{\Lambda_\gamma} = \mathbb{R} \times \Lambda_\gamma$, Λ_γ = conormal lift of geodesic γ.

McDuff domains



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The Hamiltonian flow of $H : (T^*\Sigma, \omega_\sigma) \to \mathbb{R}$, $H(q,p) = \frac{\|p\|^2}{2}$ is the **magnetic flow.**

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$$\ker \omega_\sigma|_{M_r} = \ker (r d\alpha_+ + d\alpha_-) = \ker \left(d\alpha_+ + \frac{1}{r} d\alpha_- \right),$$

with integral curves the magnetic geodesics of *geodesic curvature* $k_g \equiv 1/r$.

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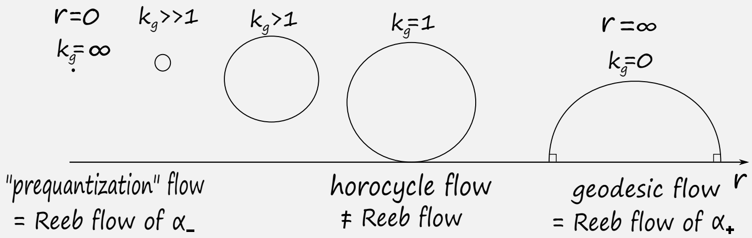

The magnetic geodesics in the universal cover $\mathbb{H}^2$.

$\mathbb{T}^2$-bundle domains

On $\mathbb{R}^3_{x,y,z}$, let

$$\alpha_\pm = \pm e^z dx + e^{-z} dy.$$

Let $A \in SL(2, \mathbb{Z}) = MCG^+(\mathbb{T}^2)$ hyperbolic, $A = diag(e^\tau, e^{-\tau})$, $\tau \neq 0$.

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Then $\alpha_\pm$ gives Liouville pair on

$$E = \mathbb{R}^3/(x,y,z) \sim (A \cdot (x,y), z - \tau),$$

a $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with hyperbolic monodromy. The pair $(\xi_+, \xi_-) = (\ker \alpha_+, \ker \alpha_-)$ supports the **suspension** of A.

Closed Lagrangians

Theorem (Cieliebak–Lazarev–Massoni–M. '22)

- ***(McDuff domains)*** *In every McDuff domain, there exist* $3g - 3$ *pairwise disjoint exact Lagrangian tori in distinct homotopy classes, where* g *denotes the genus of* Σ.
- ***(Torus bundle domains)*** *In every torus bundle domain, there are no closed exact Lagrangian submanifolds which are either orientable, projective planes, or Klein bottles.*

Idea of proof

- In the McDuff domains, the tori are Lagrangian isotopic to $\mathbb{T}^2 \cong S^*\Sigma|_\gamma$ where $\gamma \subset \Sigma$ geodesic (pair of pants decomposition $\rightsquigarrow$ $3g - 3$ geodesics).

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- In the torus bundle domains, if $L \hookrightarrow V$ is exact Lagrangian (orientable, projective plane or Klein bottle), examining the image of $\pi_1(L) \to \pi_1(V)$, L lifts to either
  - $(T^*(\mathbb{R} \times S^1), \lambda_{can})$; or
  - a subset of $(T^*\mathbb{T}^2, \lambda_{can})$ disjoint from zero section.

  But no closed exact Lagrangian in either of these (by Lalonde–Sikorav, and Gromov, respectively).

Symplectic invariants: closed strings

Theorem (Cieliebak–Lazarev–Massoni–M. '22)

$V^{2n} = [-1,1] \times M$ Liouville domain such that $M_\pm = \{\pm 1\} \times V$ is hypertight. Then:

- ***(Rabinowitz Floer cohomology)*** *We have*

$$RFH^*(V) \cong RFH^*(M_-) \oplus RFH^*(M_+),$$

  as rings.

- ***(Symplectic cohomology)*** *We have*

$$SH^*(V) = SH^*_-(V) \oplus SH^*_0(V) \oplus SH^*_+(V),$$

  as $\mathbb{Z}$-modules, with $SH^*_0(V) \cong H^*(M)$.

Fiber product structure on symplectic cohomology

Assume that free homotopy classes of orbits on M_+ are distinct from those on M_-. Then:

- $A_0 := SH_0^*$ and $A_\pm := SH_0^* \oplus SH_\pm^*$ are sub-$\mathbb{Z}$-algebras of $A := SH^*$;
- $I_\pm := SH_\pm^* \subset A_\pm \subset A$ are *ideals* such that $I_- \cap I_+ = 0$;
- $A_\pm / I_\pm \cong A_0$.

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That is, we have an algebra **fiber product** structure on symplectic cohomology:

$$\begin{array}{ccc} A & \longrightarrow & A_- \\ \downarrow & & \downarrow \\ A_+ & \longrightarrow & A_0 \end{array}$$

where maps are quotient and projection maps.

Closed string invariants of McDuff domains

Corollary (Cieliebak–Lazarev–Massoni–M. '22)

$V = \mathbb{R} \times S^*\Sigma$ McDuff domain. Then

$$SH^*(V) \cong tH^*(M)[t] \oplus H^*(M) \oplus H_{2-*}(\mathcal{L}^{nc}\Sigma),$$

where $|t| = 0$ (S^1-fibre) and $\mathcal{L}^{nc}\Sigma$ is the space of non-contractible loops on Σ.

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where $|t| = 0$ (S^1-fibre) and $\mathcal{L}^{nc}\Sigma$ is the space of non-contractible loops on Σ. As subrings:

- *$SH^*_{0-}(V) \cong H^*(M)[t]$ and $SH^*_-(V) \cong tH^*(M)[t]$ with product as polynomial rings;*
- *$SH^*_{0+}(V) \cong \check{H}^{\geqslant 0}_{2-*}(\mathcal{L}\Sigma)$ the nonnegative action part of Rabinowitz loop homology with Cieliebak–Hingston–Oancea product;*
- *$SH^*_+(V) \cong H_{2-*}(\mathcal{L}^{nc}\Sigma)$ with the loop product.*

Closed string invariants of $\mathbb{T}^2$-bundle domains

Corollary (Cieliebak–Lazarev–Massoni–M. '22)

$V = [-1,1] \times M$ a $\mathbb{T}^2$-bundle domain. Then

$$SH^*(V) \cong \bigoplus_{\Gamma} H^*(S^1) \oplus H^*(M) \oplus \bigoplus_{\Gamma} H^*(S^1),$$

where $\Gamma = \mathbb{Q} \cap [0,1)$.

Remark: Products are a bit more mysterious (they are very restricted, but contributing Floer solutions might or might not exist).

Open string products

We also have analogous fiber product descriptions for the open string products on wrapped Floer cohomology, i.e. on

$$A = \bigoplus_{\Lambda,\Lambda'} HW^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}),$$

which extends to a fiber product description of the cohomology category $H^*\mathcal{W}_0$ as

$$\begin{array}{ccc} H^*\mathcal{W}_0 & \longrightarrow & H^*\mathcal{W}_0^- \\ \downarrow & & \downarrow \\ H^*\mathcal{W}_0^+ & \longrightarrow & H^*\mathcal{W}_0^c \end{array}$$

where each category above has the same objects but different morphism groups.

McDuff domains: open string products

For the McDuff domains, consider the product $\mathfrak{m}$ on

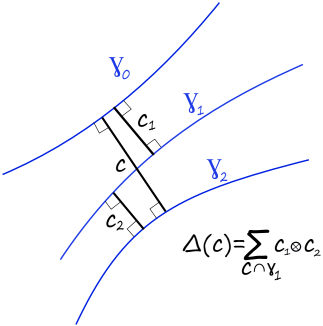
$$A = \bigoplus_{\Lambda,\Lambda'} HW^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}),$$

The ideals $I_+, I_- \subset A$ are generated by

- Binormal geodesic chords between geodesics (plus side); and
- intersection points between geodesics with positive integer multiplicities (minus side).

McDuff domains: open string products

The product $\mathfrak{m}_+$ on I_+ (geodesic side) is determined by a string co-bracket operation $\Delta : I_+ \to I_+ \otimes I_+$.



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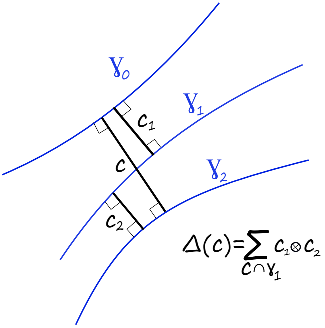
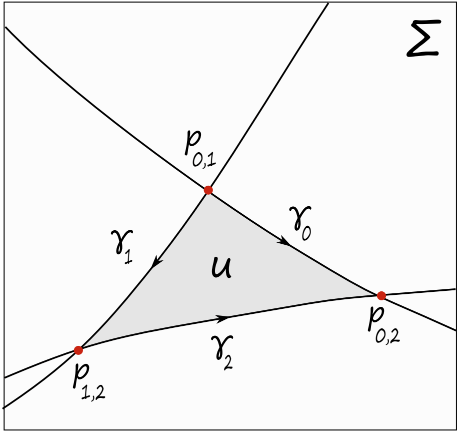


The string cobracket Δ.

The coefficients of $\mathfrak{m}_+$ are:

$$\langle \mathfrak{m}_+(c_1, c_2), c_3 \rangle = \pm\langle \Delta(c_1), c_3 \otimes \overline{c_2} \rangle \pm \langle \Delta(c_2), \overline{c_1} \otimes c_3 \rangle.$$

McDuff domains: open string products



The product $\mathfrak{m}_-$ on I_- is given by counting immersed geodesic triangles. Multiplicities at the intersection points add up.

Some open questions

(I) If two smooth Anosov flows are topologically equivalent[1], are their orbit categories quasi-equivalent? And the reciprocal?

(II) Anosov flows beyond algebraic cases are obtained e.g. by surgery. What is the effect of (Fried–Goodman) surgery on the symplectic invariants?

(III) Are the Donnay–Pugh examples of embedded surfaces in $\mathbb{R}^3$ with Anosov geodesic flow, homotopic to Anosov geodesic flows of hyperbolic metrics?

(IV) What are the symplectic invariants of the Franks–Williams example?

[1] I.e. there exists a homeomorphism sending oriented orbits to oriented orbits.

Thank you!