Symplectic methods in space mission design

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Aim of the talk: The How.

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- **(Catalogue)** Can we refine existing data bases of periodic orbits?
- **(Symplectic geometry)** Can we use modern mathematical methods from symplectic geometry to guide the numerical work?

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Preliminaries

Symplectic geometry and Hamiltonian dynamics

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A solution to the equations of motion is a curve $t \to x(t) = (q(t), p(t)) \in M$ which solves **Hamilton's equations**:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$$

Restricted three-body problem

Setup. Three massive objects: Earth (E), Moon (M), Satellite (S), under gravitational interaction.

Classical assumptions:

1. **(Restricted)** $m_S = 0$, i.e. S is *negligible.*
2. **(Circular)** The *primaries* E and M move in circles around their center of mass.
3. **(Planar)** S moves in the plane containing E and M, $\boxed{n = 2}$.

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Two parameters: μ (mass of Moon), and c (Jacobi constant = energy).

Different choices of μ models different systems in our Solar system (Jupiter–Europa, Saturn–Enceladus, etc).

Monodromy matrix

Notation: $Sp(2n) = \{\text{symplectic matrices}\}$.

- The **monodromy matrix** of a periodic orbit x is $M_x = D\phi_T^H \in Sp(2n)$, where T is the period of x, and ϕ_t^H is the Hamiltonian flow.

Note: 1 appears twice as a **trivial** eigenvalue of M_x. Can ignore them if we consider the **reduced** monodromy matrix $M_x^{red} \in Sp(2n-2)$.

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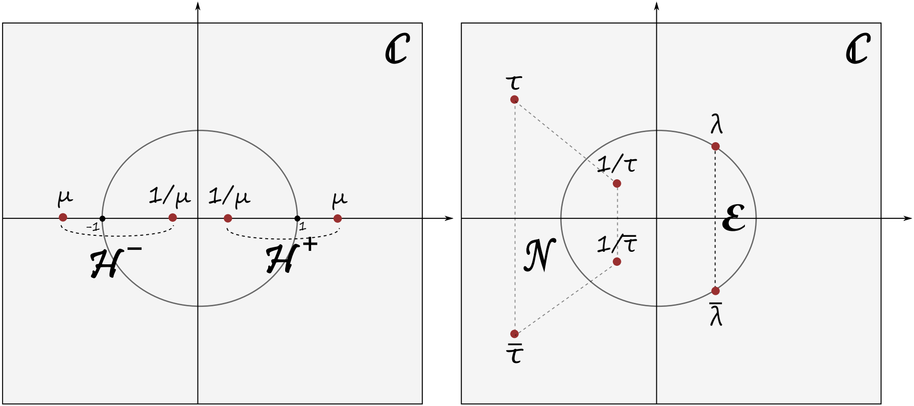
- A **Floquet multiplier** of x is an eigenvalue of M_x, which is not one of the trivial eigenvalues (i.e. an eigenvalue of M_x^{red}).
- An orbit is **non-degenerate** if 1 does not appear among its Floquet multipliers.
- An orbit is **stable** if all its Floquet multipliers are semi-simple and lie in the unit circle.

Lemma

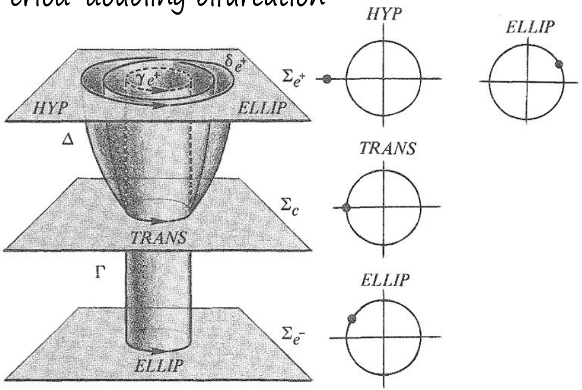
If $\mu \in \mathbb{C}$ is an eigenvalue of M_x, then so are $\overline{\mu}, 1/\mu, 1/\overline{\mu}$.



Elliptic ($\mathcal{E}$), positive/negative hyperbolic ($\mathcal{H}^{\pm}$), nonreal ($\mathcal{N}$).

Bifurcations

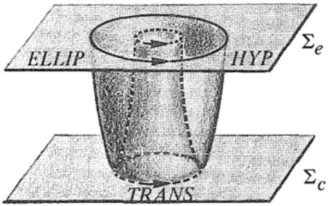
Period-doubling bifurcation

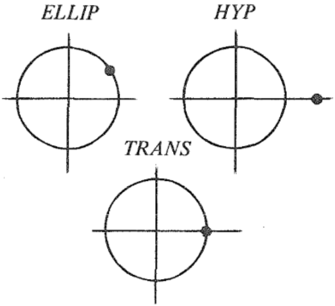


"Foundations of Mechanics", Abraham-Marsden.

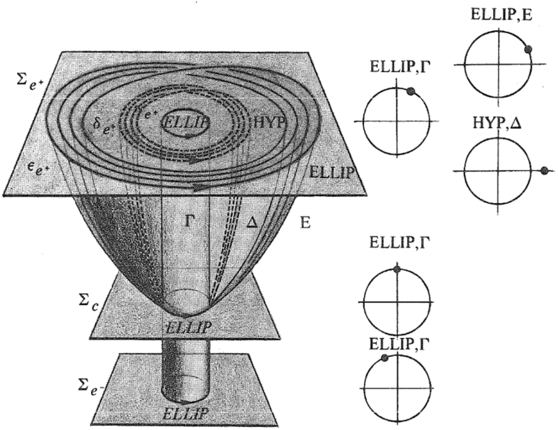
Period-doubling bifurcation or subtle division.

Bifurcations





Creation or birth/death.



Emission, or k-fold bifurcation ($k = 4$).

Symmetries

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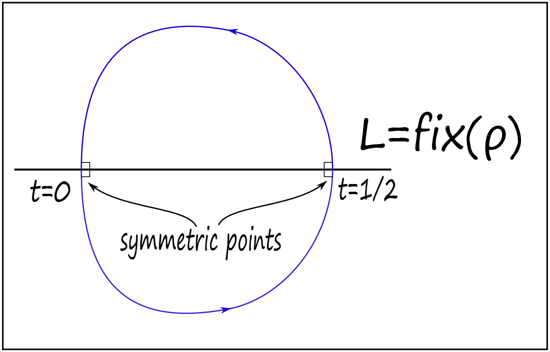
A periodic orbit x is **symmetric** if $\rho(x(-t)) = x(t)$ for all t.



Wonenburger matrices

The monodromy matrix of a symmetric orbit at a **symmetric point** has special form, a **Wonenburger** matrix:

$$M = M_{A,B,C} = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix} \in Sp(2n), \tag{1}$$

where

$$B = B^T, \quad C = C^T, \quad AB = BA^T, \quad A^T C = CA, \quad A^2 - BC = \mathbb{1}, \tag{2}$$

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The eigenvalues of M are determined by those of the first block A:

Lemma

- *λ e-val of $M \rightsquigarrow$ its **stability index** $a(\lambda) = \frac{1}{2}(\lambda + 1/\lambda)$ e-val of A.*
- *a e-val of $A \rightsquigarrow \lambda(a) = a + \sqrt{a^2 - 1}$ e-val of M.*

Toolkit

Global topological methods

These methods encode:

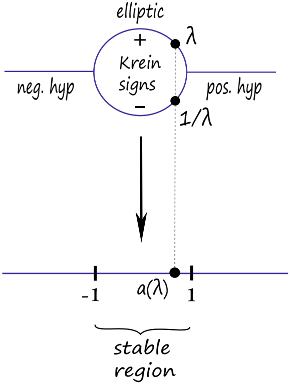
- Bifurcations;
- stability;
- eigenvalue configurations;
- obstructions to existence of regular families;
- *B-signs*,

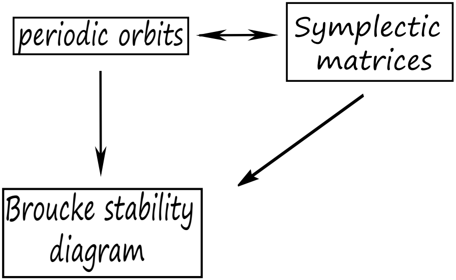
in a visual and resource-efficient way.

Broucke's stability diagram: 2D

Let $\boxed{n = 2}$, λ eigenvalue of $M^{red} \in Sp(2)$, with stability index $a(\lambda) = \frac{1}{2}(\lambda + 1/\lambda)$. Then:

- $\lambda = \pm 1$ iff $a(\lambda) = \pm 1$.
- λ positive hyperbolic iff $a(\lambda) > 1$;
- λ negative hyperbolic iff $a(\lambda) < -1$;
- λ elliptic (stable) iff $-1 < a(\lambda) < 1$.





Broucke's stability diagrams: 3D

Let $\boxed{n = 3}$. Given $M^{red} = M_{A,B,C} \in Sp(4)$, its **stability point** is $p = (\mathrm{tr}(A), \det(A)) \in \mathbb{R}^2$.

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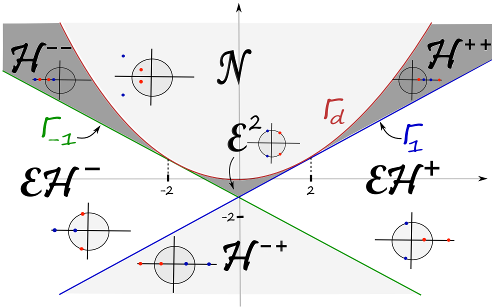
Let $\boxed{n = 3}$. Given $M^{red} = M_{A,B,C} \in Sp(4)$, its **stability point** is $p = (\mathrm{tr}(A), \det(A)) \in \mathbb{R}^2$. The plane splits into regions corresponding to the eigenvalue configuration of M^{red}:



- $\Gamma_{\pm 1}$ = e-val ± 1.
- Γ_d = double e-val.
- $\mathcal{E}^2$ = doubly elliptic (**stable region**).
- etc.

Bifurcations in the Broucke diagram

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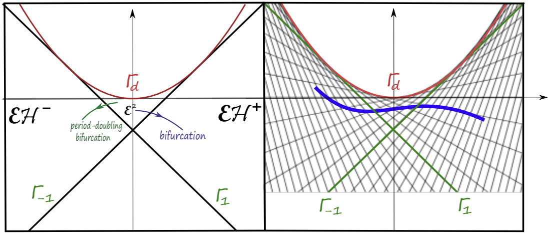

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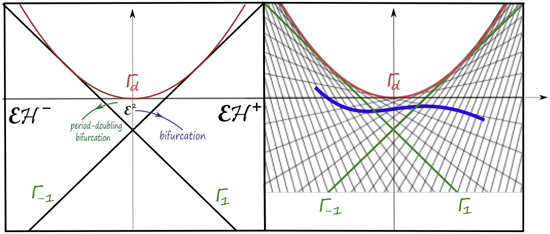

More generally:

- Γ_θ = line with slope $\cos(2\pi\theta) \in [-1, 1]$ = matrices with e-val $e^{2\pi i\theta}$;
- Γ_λ = line with slope $a(\lambda) \in \mathbb{R}\backslash[-1, 1]$ = matrices with e-val λ.

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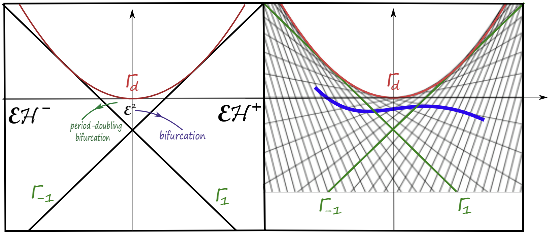

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If we know that two points lie in different components, then one should expect bifurcations in any path between them.

B-signs

Assume $\boxed{n = 2, 3}$. Let x be a symmetric orbit with monodromy

$$M_{A,B,C} = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$$

at a symmetric point. Assume a is a real, simple and nontrivial eigenvalue of A (i.e. $\lambda(a)$ elliptic or hyperbolic)). Let v satisfy $A^T v = a \cdot v$. The **B-sign** of $\lambda(a)$ is

$$\epsilon(\lambda(a)) = \text{sign}(v^T B v) = \pm.$$

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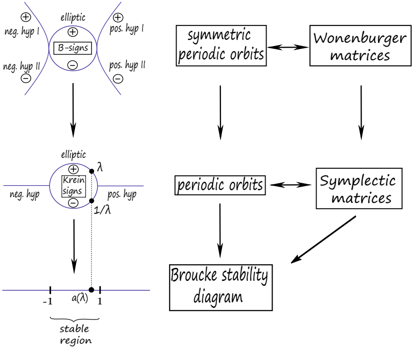
Note: Independent of v.

- $\boxed{n = 2}$ two B-signs ϵ_1, ϵ_2, one for each symmetric point.
- $\boxed{n = 3}$ two **pairs** of B-signs $(\epsilon_1^1, \epsilon_2^1), (\epsilon_1^2, \epsilon_2^2)$, one for each symmetric point and each eigenvalue.

Fact: A planar symmetric orbit is negative hyperbolic iff the B-signs of its two symmetric points differ (Frauenfelder–M. [FM], ’23).

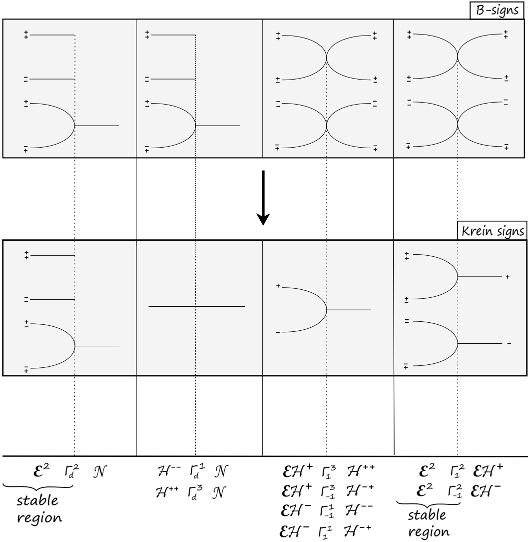
Global topological methods: GIT sequence, 2D

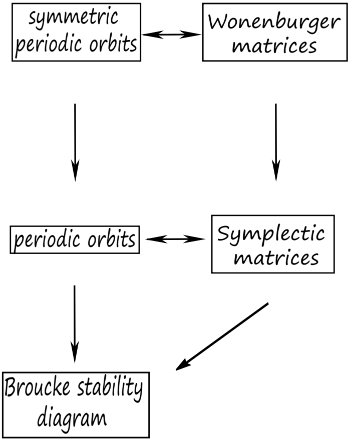
GIT sequence = refinement of Broucke diagram for **symmetric** orbits.



- *B*-signs "separate" hyperbolic branches, for symmetric orbits.
- If two points lie in the same component of the Broucke diagram, but if *B*-signs differ, one should *also* expect bifurcation in any path joining them.

Global topological methods: GIT sequence, 3D





The branches are two-dimensional, and come together at the “branching locus”, where we cross from one region to another.

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- Helps understand which families of orbits connect to which (CZ-index stays constant if no bifurcation occurs);
- Helps determine if orbits are elliptic/hyperbolic.

Conley–Zehnder index

n=2 x planar orbit with (reduced) monodromy M_x^{red}, x^k k-fold cover.

- **Elliptic case:** M_x^{red} conjugated to rotation,

$$M_x^{red} \sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

with Floquet multipliers $e^{\pm 2\pi i\theta}$. Then

$$\mu_{CZ}(x^k) = 1 + 2 \cdot \lfloor k \cdot \theta / 2\pi \rfloor$$

In particular, it is odd, and jumps by $\pm$ 2 if the e-val 1 is crossed.

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- **Hyperbolic case:**

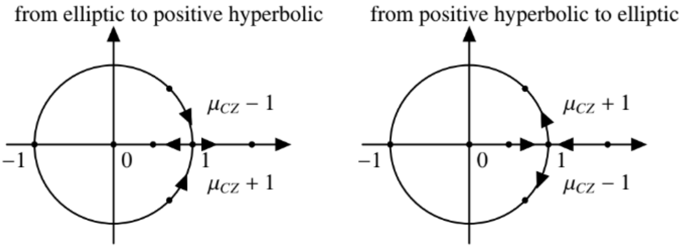
$$M_x^{red} \sim \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix},$$

  with Floquet multipliers $\lambda, 1/\lambda$. Then

$$\mu_{CZ}(x^k) = k \cdot n,$$

  where $DX_H(t)$ rotates eigenspaces by angle $\frac{\pi n t}{T}$, with n even/odd if x pos./neg. hyp.

CZ-jumps



μ_{CZ} jumps by ± 1 when crossing 1, according to direction of bifurcation. If it stays elliptic, the jump is by ± 2.

Conley–Zehnder index

$\boxed{n=3}$, planar orbits. Assume H admits the reflection along the (x, y)-plane as symmetry (e.g. 3BP). If $x \subset \mathbb{R}^2$ planar orbit,

$$M_x^{red} \sim \begin{pmatrix} M_p^{red} & 0 \\ 0 & M_s \end{pmatrix} \in Sp(4).$$

Then

$$\mu_{CZ}(x) = \mu_{CZ}^p(x) + \mu_{CZ}^s(x),$$

where each summand corresponds to M_p^{red} and M_s respectively.

- Planar to planar bifurcations correspond to jumps in μ_{CZ}^p.
- Planar to spatial bifurcations correspond to jumps of μ_{CZ}^s.

Floer numerical invariants

- A periodic orbit x is **good** if $\mu_{CZ}(x^k) = \mu_{CZ}(x) (\text{mod } 2)$ for all $k \geqslant 1$.

Note: a planar orbit is bad iff it is an even cover of a negative hyperbolic orbit.

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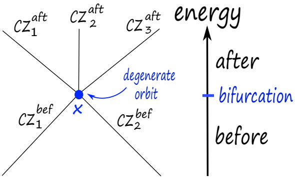


Given a bifurcation at x, the *SFT-Euler characteristic* (or the *Floer number*) of x is

$$\chi_{SFT}(x) = \sum_i (-1)^{CZ_i^{bef}} = \sum_j (-1)^{CZ_j^{aft}}.$$

The sum on the LHS is over **good** orbits *before* bifurcation, and RHS is over **good** orbits *after* bifurcation.

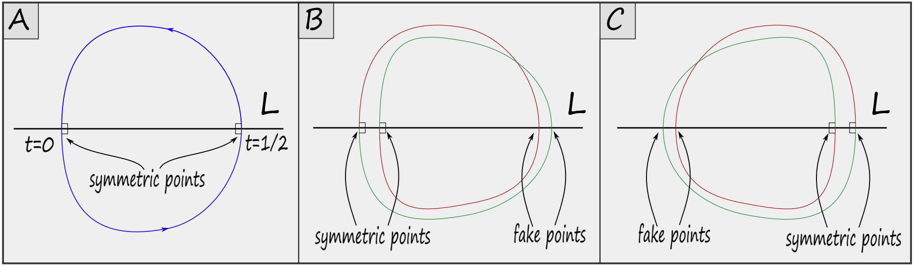
Invariance

The fact that the sums agree before and after –*invariance*– follows from **Floer theory** in symplectic geometry.

In Memoriam Andreas Floer, 1956-1991.

The Floer number can be used as a **test**: if the sums do *not* agree, we know the algorithm missed an orbit.

Example: symmetric period doubling bifurcation



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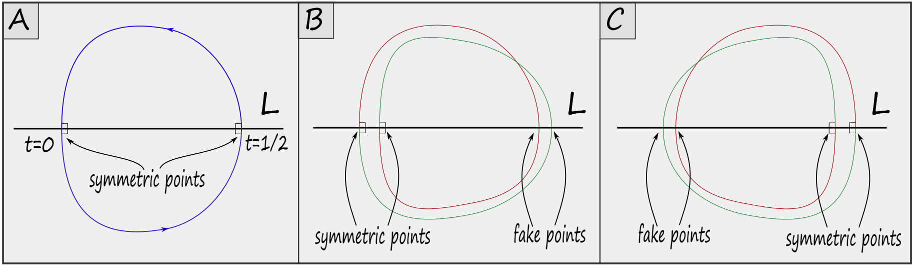

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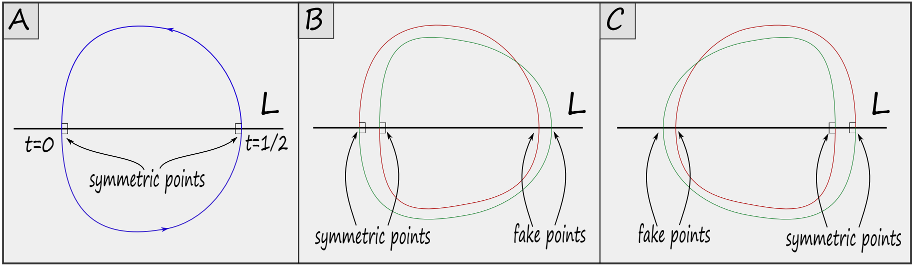

The simple symmetric orbit x goes from elliptic to negative hyperbolic.

- A priori there could be two bifurcations for each symmetric point (B or C).
- Invariance of $\chi_{SFT}(x^2)$ implies only one can happen (note x^2 is *bad*).
- Bifurcation happens at the symmetric point in which the B-sign does *not* jump.

Summary of toolkit

(1) **The B-signs**: a number associated to each elliptic or hyperbolic Floquet multiplier of an orbit, which helps predict bifurcations.

(2) **Global topological methods:** the *GIT-sequence*, a topological refinement of *Broucke's stability diagram*, which encodes bifurcations and stability of orbits.

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Numerical work

Missions

To find conditions suitable for life, missions proposed by **NASA**:

- Jupiter-Europa system (Europa Clipper); and
- Saturn-Enceladus system.

This motivates studies of orbits for these systems.

The power of deformations

Two options:

- Fix μ and change c; or
- Fix c and change μ.

The power of deformations

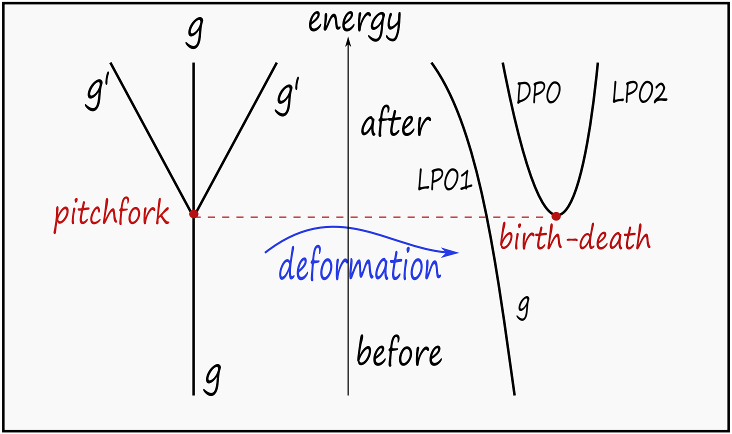
Two options:

- Fix μ and change c; or
- Fix c and change μ.

I.e. to study a system, sometimes it is worthy to study another nearby system:

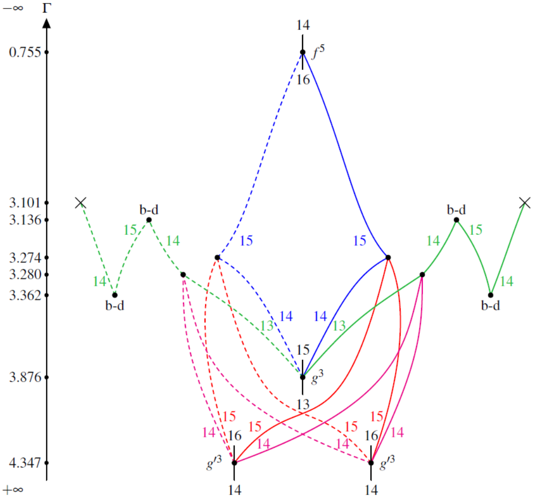
Hill's lunar problem $\rightsquigarrow$ Saturn–Enceladus $\rightsquigarrow$ Jupiter–Europa $\rightsquigarrow$ Earth–Moon.

Example: Pitchfork bifurcation



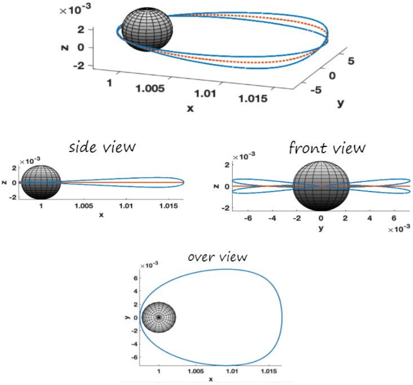
Lunar problem has more symmetry: a (non-generic) pitchfork bifurcation in lunar problem (Hénon) deforms to a generic situation in Jupiter–Europa. Birth-death branch might be hard to predict otherwise.

Hill's lunar problem



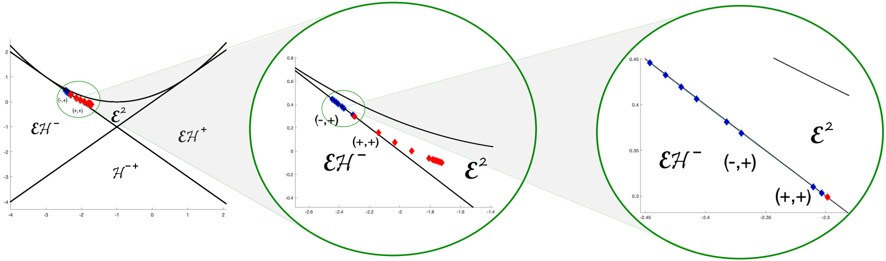
Bifurcation diagram involving covers of f, g, g' (Cengiz Aydin, PhD thesis '23). Each family has constant CZ-index. Floer invariants are easy to compute.

Numerical work

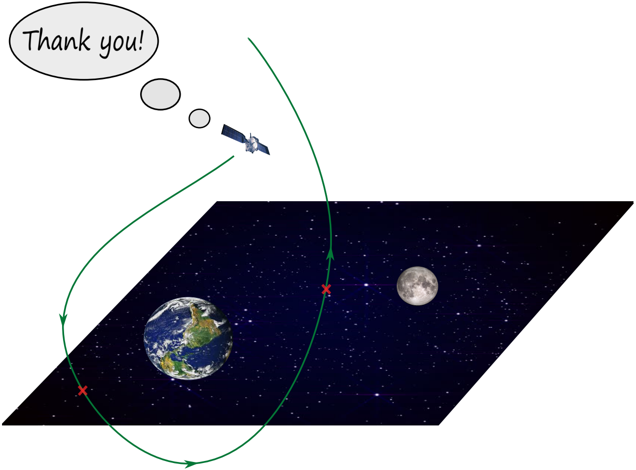


Period-doubling bifurcation in the Jupiter-Europa system ($\mu = 2.5266448850435E^{-05}$), found via the *cell-mapping method* of Koh–Anderson–Bermejo-Moreno [KAB].

GIT plots



GIT plot of the period-doubling bifurcation of the snitch configuration (Frauenfelder–Koh–M. [FKM]).


Thank you!

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