On the spatial restricted three-body problem

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Spin-off (holomorphic dynamics): arXiv:2011.06568.
Survey available: arXiv:2101.04438.

Spatial circular restricted three-body problem

Setup. Three objects: Earth (E), Moon (M), Satellite (S) with masses m_E, m_M, m_S, under gravitational interaction.

Classical assumptions:

1. **(Restricted)** $m_S = 0$, i.e. S is *negligible.*
2. **(Circular)** The *primaries* E and M move in circles around their center of mass.
3. **(Planar)** S moves in the plane spanned by E and M.

Spatial case: drop the planar assumption.

Goal: Study motion of S.

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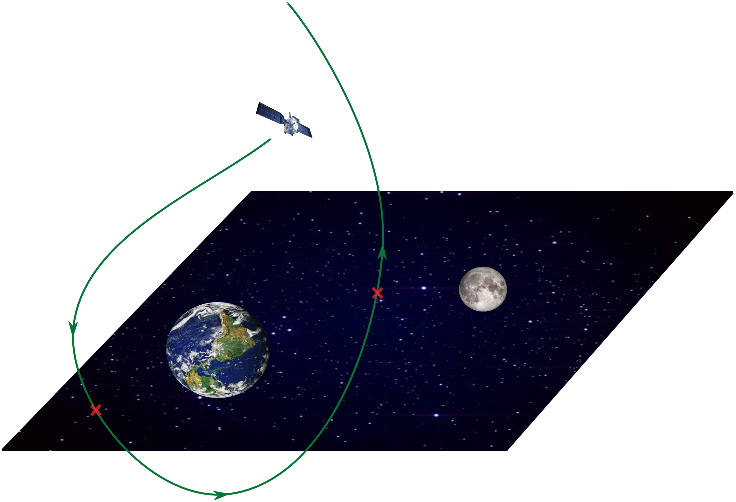

Spatial circular restricted three-body problem

In rotating coordinates so that E, M are fixed, the Hamiltonian is autonomous and so a conserved quantity:

$$H : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}$$

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q-M\|} - \frac{1-\mu}{\|q-E\|} + p_1 q_2 - p_2 q_1,$$

where we normalize so that $m_E + m_M = 1$, and $\mu = m_M$.

Planar problem: $p_3 = q_3 = 0$ (flow-invariant subset).

Two parameters: μ, and $H = c$ Jacobi constant.

Integrable limit cases

If $\mu = 0 \rightsquigarrow H = K + L$, where

$$K(q,p) = \frac{1}{2}\|p\|^2 - \frac{1}{\|q\|}$$

is the *Kepler energy* (two-body problem), and

$$L = p_1 q_2 - p_2 q_1$$

is the Coriolis/centrifugal term. This is the *rotating Kepler problem.*

Fact: $c \to -\infty \rightsquigarrow$ Kepler problem.

Hill regions

H has five critical points: $L_1, \ldots, L_5$ called *Lagrangians*.

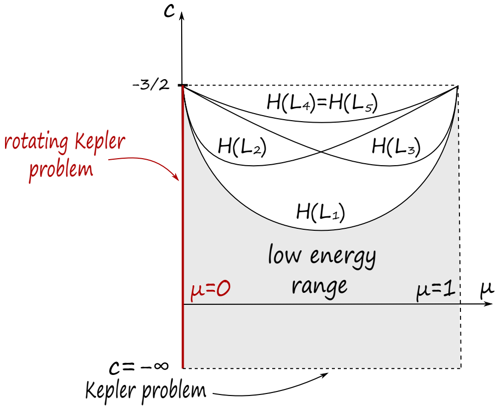

Figure: The critical values of H.

Hill regions

For $c \in \mathbb{R}$, let $\Sigma_c = H^{-1}(c)$. Consider

$$\pi : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}^3 \backslash \{E, M\}$$

$$(q, p) \mapsto q,$$

and the *Hill region*

$$\mathcal{K}_c = \pi(\Sigma_c).$$

Low energy Hill regions

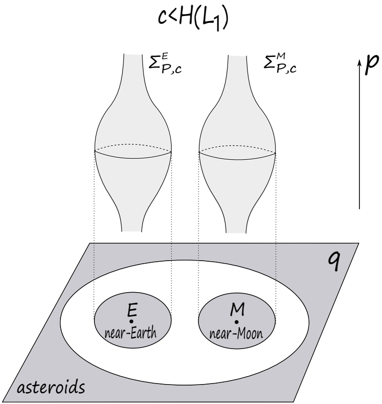

Figure: Morse theory in the three-body problem.

Low energy Hill regions

$c \in (H(L_1), H(L_1) + \varepsilon)$

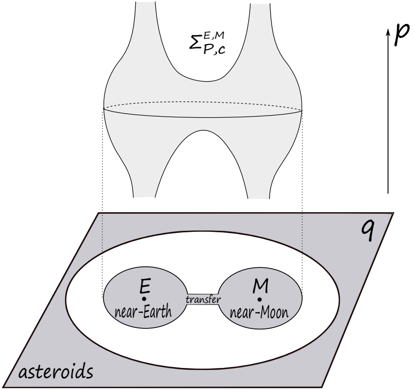

Figure: Morse theory in the three-body problem.

Moser regularization

H is singular at *collisions* ($q = E$ or $q = M \rightsquigarrow p = \infty$), but can be regularized via Moser's recipe:

$$(q, p) \overset{\text{switch}}{\longmapsto} (-p, q) \overset{\text{stereo. proj.}}{\longmapsto} (\xi, \eta) \in T^*S^3$$

We get compactifications for spatial energy levels:

$$\Sigma_c^E \rightsquigarrow \overline{\Sigma}_c^E \cong S^*S^3.$$

$$\Sigma_c^M \rightsquigarrow \overline{\Sigma}_c^M \cong S^*S^3.$$

$$\Sigma_c^{E,M} \rightsquigarrow \overline{\Sigma}_c^{E,M} \cong S^*S^3 \# S^*S^3.$$

Similarly, the planar problem level sets get compactified to $\overline{\Sigma}_{P,c}^E \cong S^*S^2 = \mathbb{R}P^3$, $\overline{\Sigma}_{P,c}^M \cong S^*S^2 = \mathbb{R}P^3$, $\overline{\Sigma}_{P,c}^{E,M} \cong \mathbb{R}P^3 \# \mathbb{R}P^3$.

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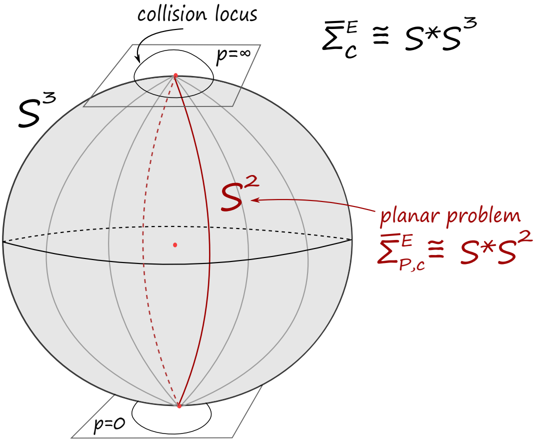

Figure: The Moser-regularized level set near E.

Contact geometry of the three-body problem

Theorem (planar case: Albers-Frauenfelder-van Koert-Paternain '12, spatial case: Cho-Jung-Kim '19)

For $\mu \in (0,1)$, $c < H(L_1)$, $\overline{\Sigma}_c^E$ and $\overline{\Sigma}_c^M$ are contact-type, and so is $\overline{\Sigma}_c^{E,M}$ for $c \in (H(L_1), H(L_1)+\epsilon)$ for some $\epsilon > 0$. As contact manifolds:

$$\overline{\Sigma}_c^E \cong \overline{\Sigma}_c^M \cong (S^*S^3, \xi_{std}),$$

$$\overline{\Sigma}_c^{E,M} \cong (S^*S^3, \xi_{std}) \# (S^*S^3, \xi_{std}).$$

The planar problem is a flow-invariant codim-2 contact submanifold:

$$\overline{\Sigma}_{P,c}^E \cong \overline{\Sigma}_{P,c}^M \cong (S^*S^2, \xi_{std}),$$

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Poincaré-Birkhoff and the planar problem

In his long search for closed orbits in the planar three-body problem, Poincaré's approach can be reduced to:

(1) Finding a global surface of section for the dynamics;
(2) Proving a fixed point theorem for the arising return map.

This is the setting for Poincaré-Birkhoff's theorem:

An area-preserving homeomorphism of an annulus that rotates the two boundaries in opposite directions (the twist condition) has at least two fixed points.

Goal: Generalize this approach to the *spatial* problem.

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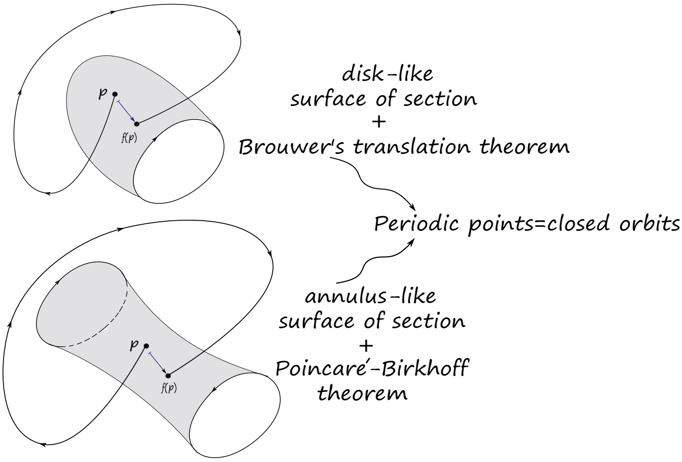

Figure: Obtaining closed orbits.

Open book decompositions

An *open book decomposition* on a closed odd-dimensional manifold M is a fibration $\pi : M \backslash B \to S^1$, where $B \subset M$ is a closed codimension-2 submanifold with trivial normal bundle, and $\pi(b, r, \theta) = \theta$ on some collar neighbourhood $B \times \mathbb{D}^2$ of B.

Abstract data: *page* $P = \overline{\pi^{-1}}(pt)$ (with $B = \partial P$ *binding*), *monodromy* $\phi : P \stackrel{\cong}{\to} P$, $\phi|_B = id$.

$$(P, \phi) \rightsquigarrow M = \mathbf{OB}(P, \phi) = P_\phi \bigcup B \times \mathbb{D}^2,$$

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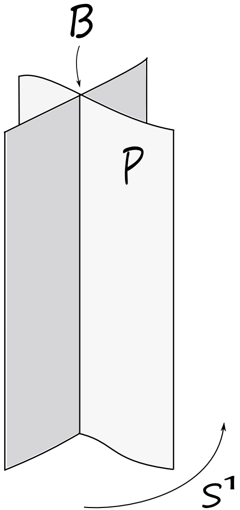

Global hypersurfaces of section

If $\varphi_t : M \to M$ is a flow on M generated by an autonomous vector field X, then π is *adapted to the dynamics* if B is φ_t-invariant (i.e. $X|_B$ is tangent to B), and X is transverse to the interior of all pages.

Each page P is a *global hypersurface of section*, i.e. it is codimension-1, $B = \partial P$ is a union of orbits, and the orbits of all points in $M \backslash B$ meet the interior of each page transversely in the future and past.

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Open books and surfaces of section in the planar problem: historical remarks

Planar situation: smoothly $\mathbb{R}P^3 = \mathbf{OB}(\mathbb{D}^*S^1, \tau_0^2)$, where $\tau_0 =$Dehn twist along $S^1 \subset \mathbb{D}^*S^1$.

Perturbative methods:

- If $\mu \sim 0$ is small and $c < H(L_1)$, Poincaré [P12] provides annulus-like global surfaces of section by perturbing the rotating Kepler problem.
- If $c \ll H(L_1)$ and $\mu \in (0,1)$, Conley [C63] shows there are annulus-like surfaces of section and the return map is a Birkhoff twist map, and uses Poincaré-Birkhoff.
- McGehee [M69] provides a disk-like global surface of section for $\mu \sim 0$ small and $c < H(L_1)$, and computes the return map.

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convexity range: $\mathcal{C} = \{(\mu, c),\ c < H(L_1)$: Levi-Civita regularization of planar problem is convex$\}$.

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Step 1: Open books in the spatial three-body problem

$\overline{\Sigma}_c = H^{-1}(c)$ compact and connected component of a (regularized) energy hypersurface in the SCR3BP.

Theorem (M.–van Koert)

For $\mu \in (0,1)$, we have

$$\overline{\Sigma}_c = \begin{cases} \boldsymbol{OB}(\mathbb{D}^* S^2, \tau^2), & \textit{if } c < H(L_1) \\ \boldsymbol{OB}(\mathbb{D}^* S^2 \natural \mathbb{D}^* S^2, \tau_1^2 \circ \tau_2^2), & \textit{if } c \in (H(L_1), H(L_1)+\epsilon), \end{cases}$$

which are adapted to the dynamics. Here, τ is the Dehn-Seidel twist along the zero section $S^2 \subset \mathbb{D}^* S^2$.

Binding $B = S^* S^2 = \partial \mathbb{D}^* S^2 = \mathbb{R}P^3 =$ planar problem for energy c.

Step 1: Open books in the spatial three-body problem

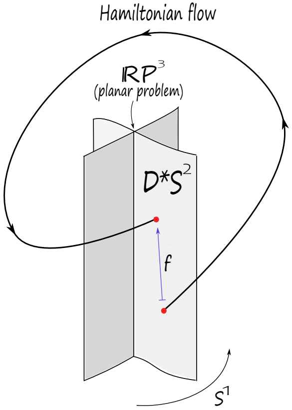

Figure: The open book in the spatial problem for $c < H(L_1)$.

Basic idea

Let $B = \{p_3 = q_3 = 0\}$ (planar problem). Define

$$\pi(q,p) = \frac{q_3 + ip_3}{\|q_3 + ip_3\|} \in S^1, \; d\pi = \frac{p_3 dq_3 - q_3 dp_3}{p_3^2 + q_3^2}.$$

Then

$$d\pi(X_H) = \frac{p_3^2 + q_3^2 \cdot \left(\frac{1-\mu}{\|q-E\|^3} + \frac{\mu}{\|q-M\|^3}\right)}{p_3^2 + q_3^2} > 0,$$

for $p_3^2 + q_3^2 \neq 0$, and numerator vanishes only along B.

Problem: This does not extend to the collision locus.

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Physical interpretation

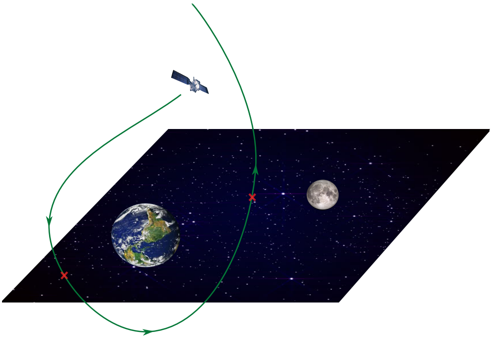

Figure: The $\pi/2$-page corresponds to $q_3 = 0$, $p_3 > 0$, and means that the spatial orbits of S are transverse to the plane spanned by E, M away from collisions.

Polar orbits

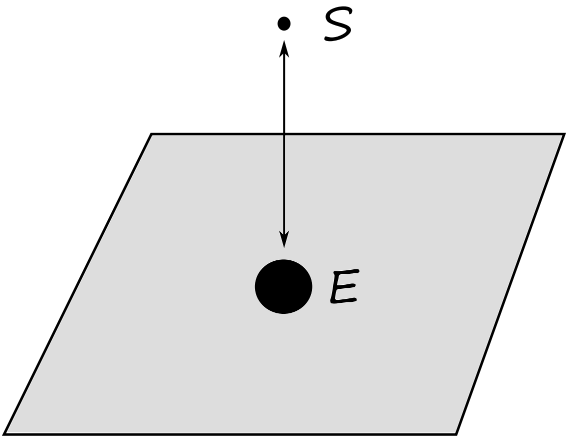

Figure: Polar orbits prevent transversality on the collision locus.

Return map

Theorem (M.–van Koert)

For every $\mu \in (0,1]$, $c < H(L_1)$, and page P, the return map f extends smoothly to the boundary $B = \partial P$, and in the interior it is an exact symplectomorphism

$$f = f_{c,\mu} : (int(P), \omega) \to (int(P), \omega),$$

where $\omega = d\alpha|_P$, $\alpha = \alpha_{\mu,c}$ contact form. Moreover, f is Hamiltonian in the interior, and the Hamiltonian isotopy extends smoothly to the boundary.

Here, ω degenerates at B, but after a continuous conjugation, it is *deformation equivalent* to the standard symplectic form. The Hamiltonian is *not* rel boundary.

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Spatial vs Planar orbits

Note that

$$\mathrm{Fix}(f^k) = \mathrm{IntFix}(f^k) \bigcup \mathrm{BdyFix}(f^k),$$

where

$$\mathrm{IntFix}(f^k) \longleftrightarrow \{\text{spatial orbits of period } k\}$$

$$\mathrm{BdyFix}(f^k) \to \{\text{planar orbits}\}$$

Goal: Find *interior* periodic points with arbitrary large minimal k.

Step 2: Fixed point theory of Hamiltonian twist maps

$(W, \omega = d\lambda)$ Liouville domain, $\alpha = \lambda|_B$. Let $f : (W, \omega) \to (W, \omega)$ be a Hamiltonian symplectomorphism.

Definition

f is a *Hamiltonian twist map* if there exists a time-dependent Hamiltonian $H : \mathbb{R} \times W \to \mathbb{R}$ such that:

- H is *smooth* (or C^2);
- $f = \phi_H^1$;
- There exists a smooth function $h : \mathbb{R} \times B \to \mathbb{R}$ which is *positive* and
$$X_{H_t}|_B = h_t R_\alpha.$$

Fixed-point theorem

Theorem (M.–van Koert, Generalized Poincaré–Birkhoff theorem)

Suppose that f is an exact symplectomorphism of a Liouville domain (W, λ), *and let* $\alpha = \lambda|_B$. *Assume the following:*

- ***(Hamiltonian twist map)*** *f is a Hamiltonian twist map;*
- ***(index-definiteness)*** *If* $\dim W \geq 4$, *then assume* $c_1(W)|_{\pi_2(W)} = 0$, *and* $(\partial W, \alpha)$ *is strongly index-definite. In addition, assume all fixed points of f are isolated;*
- ***(Symplectic homology)*** $SH_\bullet(W)$ *is infinite dimensional.*

Then f has simple interior periodic points of arbitrarily large (integer) period.

A few remarks

- Strong index definiteness is a technical assumption, implied by strict convexity.
- If dim $W = 2$, dim $SH_\bullet(W) = \infty$ iff $W \neq \mathbb{D}^2$.
- A very vast generalization of the classical Poincaré-Birkhoff theorem, in the spirit of the Conley conjecture (good).
- We couldn't check the twist condition in the three-body problem (not so good).

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Holomorphic dynamics

Observation: the adapted open book $\mathbf{OB}(\mathbb{D}^*S^2, \tau^2)$ is *iterated planar* (IP), i.e. the page $\mathbb{D}^*S^2 = \mathbf{LF}(\mathbb{D}^*S^1, \tau_P^2)$ admits a Lefschetz fibration with genus zero fibers, all inducing the open book $\mathbf{OB}(\mathbb{D}^*S^1, \tau_P^2)$ at the binding $\mathbb{R}P^3$.

$T^*S^2 = \mathbf{LF}(T^*S^1, \tau_P^2)$

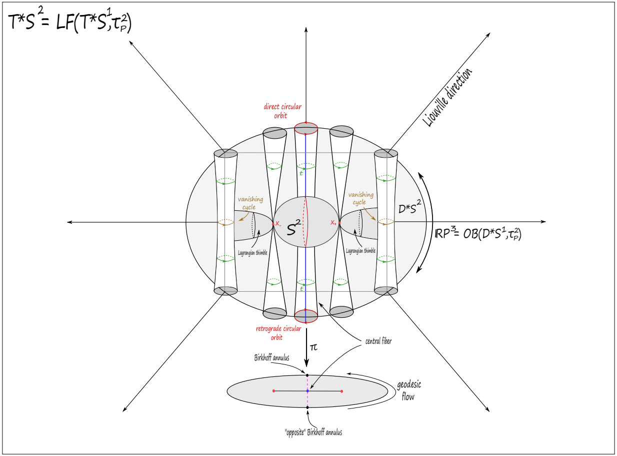

Figure: The standard Lefschetz fibration on T^*S^2.

Abstract page

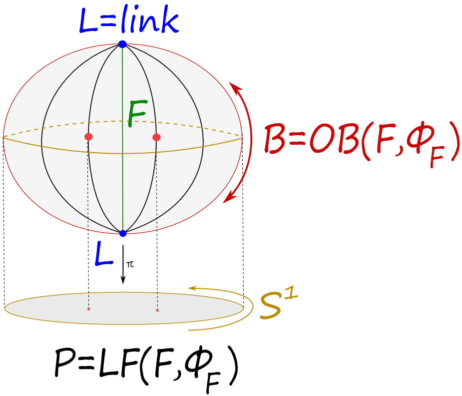

Figure: Abstractly, the compact version of the Lefschetz fibration on a page P. F is the regular fiber, $L = \partial F$ is the "binding of the binding" B, a link.

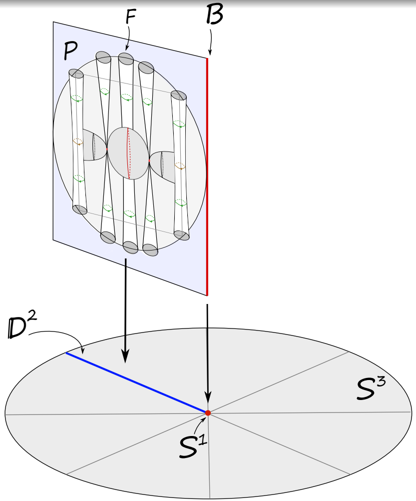

Figure: The moduli space of fibers is a copy of $S^3 = \mathbf{OB}(\mathbb{D}^2, \mathbb{1})$.

Contact structures and Reeb dynamics on moduli

Let $(M, \xi_M) = \textbf{OB}(P, \phi)$ be an IP 5-fold, $P = \textbf{LF}(F, \phi_F)$.

$\textbf{Reeb}(P, \phi) = \{\alpha$ adapted contact form: $\alpha|_B$ adapted to $B = \textbf{OB}(F, \phi_F)\}$.

Theorem (M., Contact structures and Reeb dynamics on moduli)

For a given $\alpha \in \textbf{Reeb}(P, \phi)$, there is a moduli space $\mathcal{M}$ of $d\alpha$-symplectic copies of F foliating M, forming the fibers of a Lefschetz fibration on each page. $\mathcal{M}$ is a contact manifold $(\mathcal{M}, \xi_{\mathcal{M}}) \cong (S^3, \xi_{std}) = \textbf{OB}(\mathbb{D}^2, \mathbb{1})$.

Any $\alpha \in \textbf{Reeb}(P, \phi)$ induces a contact form $\alpha_{\mathcal{M}} \in \textbf{Reeb}(\mathbb{D}^2, \mathbb{1})$, $\ker \alpha_{\mathcal{M}} = \xi_{\mathcal{M}}$, adapted to a trivial open book of the form $\theta_{\mathcal{M}} : \mathcal{M} \backslash \mathcal{M}_B \cong S^3 \backslash S^1 \to S^1$.

Idea: fiber-wise integration

The contact form $\alpha_{\mathcal{M}}$ is defined via

$$(\alpha_{\mathcal{M}})_u(v) = \int_{z \in F_u} \alpha_z(v(z))dz,$$

where $F_u = \mathrm{im}(u)$, $dz = d\alpha|_{F_u}$, $u \in \mathcal{M}$, $v \in T\mathcal{M}$. Its Reeb vector field $R_{\mathcal{M}}$ is defined via

$$\mathbf{D}_u R_{\mathcal{M}} = 0, \text{ where } \mathbf{D}_u = \text{ linearized CR-operator},$$

$$1 = (\alpha_{\mathcal{M}})_u(R_{\mathcal{M}}(u)) = \int_{z \in F_u} \alpha_z(R_{\mathcal{M}}(z))dz,$$

$$0 = (d\alpha_{\mathcal{M}})_u(R_{\mathcal{M}}(u), \cdot) = \int_{z \in F_u} d\alpha_z(R_{\mathcal{M}}(z), \cdot)dz.$$

$R_{\mathcal{M}}$ is a reparametrization of an L^2-projection of R_α to $T\mathcal{M}$.

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Return map and symplectic tomographies

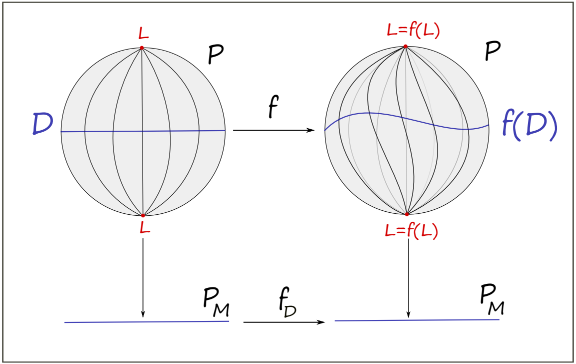

Figure: The return map f might not preserve the symplectic foliation. One can take *symplectic tomographies D* (a symplectic 2-disk) to induce return maps f_D on $\mathcal{M}$.

Shadowing cone

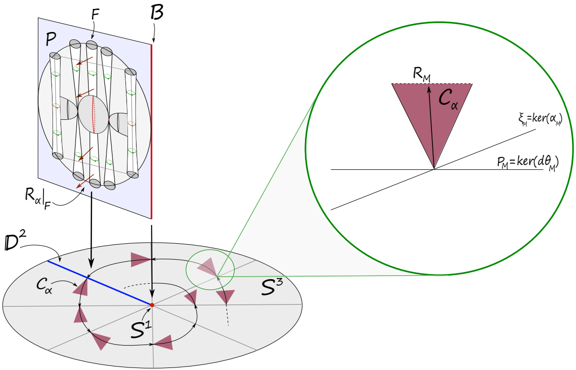

Figure: The shadowing cone is $C_\alpha = \pi_*(\ker d\alpha)$. Orbits of α project to orbits of the cone, which are transverse to $\xi_{\mathcal{M}}$ and to every page. The Reeb vector field $R_{\mathcal{M}}$ spans the average direction of C_α.

Holomorphic shadow

Define the *holomorphic shadow map* as

$$\textbf{HS} : \textbf{Reeb}(P, \phi) \to \textbf{Reeb}(\mathbb{D}^2, \mathbb{1})$$

$$\alpha \mapsto \alpha_{\mathcal{M}}.$$

Integrable case: Rotating Kepler problem $\mapsto$ Hopf flow on S^3.

The return map preserves the foliation. The two nodal singularities are fixed, and correspond to the polar orbits. The map is a classical twist map on the annuli fibers.

Theorem (M., Reeb lifting theorem)

HS *is surjective.*

In other words, Reeb dynamics in M is at least as complex as Reeb dynamics in S^3.

New program: Try to "lift" knowledge from dynamics on S^3.

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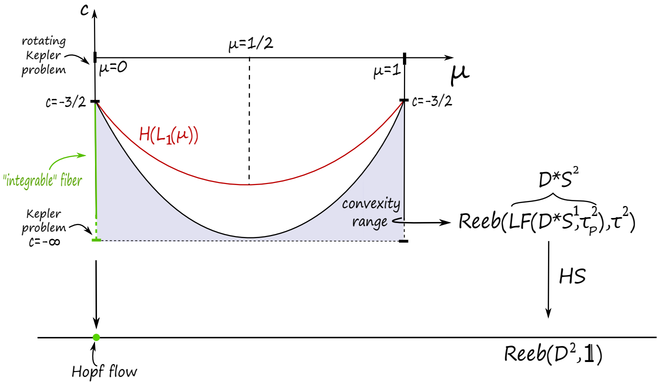
Case of three-body problem

If $(\mu, c) \in \mathcal{C}$, combining our adapted open book with [HSW] on $B = \mathbb{R}P^3 \rightsquigarrow \alpha_{\mu,c} \in \textbf{Reeb}(\mathbb{D}^* S^2, \tau^2)$.



Further directions: Entropy

Joint work in progress with Umberto Hrynewicz, Abror Pirnapasov:

Claim 1: C^∞-generic Reeb flows on any closed 3-fold have positive topological entropy.

Pull back via the shadow map $\rightsquigarrow$

Claim 2: C^∞-generic Reeb flows in $\mathbf{Reeb}(P, \phi)$ also have positive topological entropy, for every IP 5-fold, generated by purely spatial orbits.

Closing remarks

- Hamiltonian maps which are not the identity at the boundary should perhaps be studied more systematically, specially in higher dimensions.
- The Hamiltonian twist condition, if true at all, seems HARD to check.
  Enter the famous Katok examples! they are a counterxample to the conclusion of the theorem, i.e. they are not twist maps. BUT they are arbitrarily close to the Kepler problem (geodesic flow on S^3).
- This is a good time to revisit the origins.

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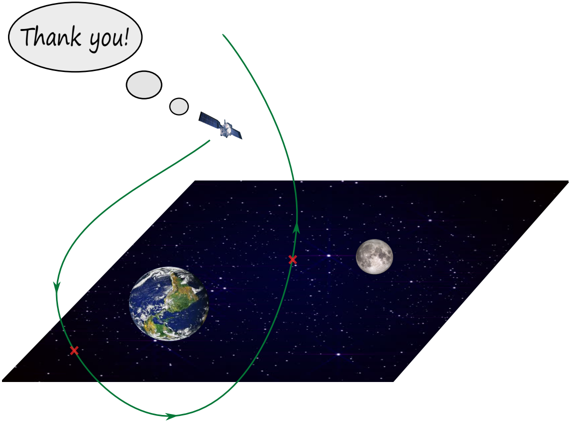

Thank you!

Complementary slides: Index growth

We call a strict contact manifold $(Y, \xi = \ker \alpha)$ *strongly index-definite* if the contact structure $(\xi, d\alpha)$ admits a symplectic trivialization ϵ so that:

- There are constants $c > 0$ and $d \in \mathbb{R}$ such that for every Reeb chord $\gamma : [0, T] \to Y$ of Reeb action $T = \int_0^T \gamma^*\alpha$ we have
$$|\mu_{RS}(\gamma; \epsilon)| \geq cT + d,$$
where μ_{RS} is the Robbin–Salamon index.

Drop absolute value $\rightsquigarrow$ index-*positive*.

Complementary slides: Examples of index-positivity

Lemma (Some examples)

- *If $(Y, \alpha) \subset \mathbb{R}^4$ is a strictly convex hypersurface, then it is strongly index-positive.*
- *If $(Y, \ker \alpha) = (S^*Q, \xi_{std})$ is symplectically trivial and (Q, g) has positive sectional curvature, then (Y, α) is strongly index-positive.*

Complementary slides: special case of fixed-point theorem

Theorem (M.–van Koert, special case)

Let $W \subset (T^*M, \lambda_{can})$ be fiber-wise star-shaped, with M simply connected, orientable and closed. Let $f : W \to W$ be a Hamiltonian twist map. Assume:

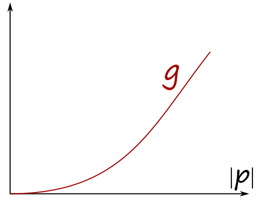
- *Reeb flow on ∂W is strongly index-positive; and*
- *All fixed points of f are isolated.*

Then f has simple interior periodic points of arbitrarily large period.

Complementary slides: Toy example

$Q = S^n$ with round metric.
$H : T^*Q \to \mathbb{R}$, $H(q,p) = 2\pi|p|$ *not* smooth at zero section.
Then $\phi_H^1 = id$, all orbits are periodic with same period.

Let $K = 2\pi g$, with $g = g(|p|)$ smoothing of $|p|$ near $p = 0$. Then $\phi_K^1 = \phi_G^{2\pi g'(|p|)}$, where ϕ_G^t geodesic flow, is a Hamiltonian twist map. It has simple orbits of arbitrary period ($g'(|p|) = l/k$ coprime $\rightsquigarrow$ k-periodic orbit).



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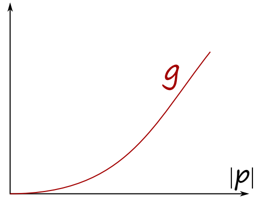

Complementary slides: dynamical applications

Definition

Let P be a page, and $f : \text{int}(P) \to \text{int}(P)$ a return map. A *fiber-wise* k-recurrent point is $x \in \text{int}(P)$ such that $f(\mathcal{M}_x) \cap \mathcal{M}_x \neq \emptyset$.

This is a "symplectic version" of a leaf-wise intersection.

Theorem (M.)

In the SCR3BP, for every k, one can find sufficently small perturbations of the integrable cases which admit infinitely many fiber-wise k-recurrent points.

More further directions: Lagrangians

Conjecture (Long interior chords)

Suppose that f is an exact symplectomorphism of a Liouville domain (W, λ), let $\alpha = \lambda|_B$, and $L \subset (W, \lambda)$ exact, spin, Lagrangian with Legendrian boundary. Assume the following:

- ***(Hamiltonian twist map)*** *f is a Hamiltonian twist map;*
- ***(index-definiteness)*** *If $\dim W \geq 4$, then assume $c_1(W)|_{\pi_2(W)} = 0$, and $(\partial W, \alpha)$ is strongly index-definite;*
- ***(Wrapped Floer homology)*** *$WFH_\bullet(L)$ is infinite dimensional.*

Then $f^k(int(L)) \cap int(L)$ is non-empty for k arbitrarily large.

Motivation: Finding long spatial collision orbits in the 3BP.

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